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Analysis of an adjoint problem approach to the identification of an unknown diffusion coefficient

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Abstract

An inverse problem for the identification of an unknown coefficient in a quasilinear parabolic partial differential equation is considered. We present an approach based on utilizing adjoint versions of the direct problem in order to derive equations explicitly relating changes in inputs (coefficients) to changes in outputs (measured data). Using these equations it is possible to show that the coefficient to data mappings are continuous, strictly monotone and injective. The equations are further exploited to construct an approximate solution to the inverse problem and to analyse the error in the approximation. Finally, the results of some numerical experiments are displayed.

1. Introduction

Using partial differential equations to model physical systems is one of the oldest activities in applied mathematics. A complete model requires certain state inputs in the form of initial and/or boundary data together with what might be called structure inputs such as coefficients or source terms which are related to the physical properties of the system. Obtaining a unique solution for the associated well-posed problem constitutes what we will call solving the direct problem. Solving the direct problem permits the computation of various system outputs of physical interest. On the other hand, when some of the required inputs are not available we may instead be able to determine the missing inputs from outputs that are measured rather than computed by formulating and solving an appropriate inverse problem. In particular, when the missing inputs are one or more unknown coefficients in the partial differential equation, the problem is called a coefficient identification problem. The identification of a diffusion coefficient in a quasilinear diffusion equation is chosen here as a prototype coefficient identification problem that has been approached by various methods.

The most common technique for identifying an unknown coefficient from some measured output is the method of output least squares (OLS) [1, 4, 8–10]. Here the unknown coefficient,

C, is chosen from an appropriate space K and the output, $\Phi[C]$, is computed by solving the direct problem. One defines an error functional, $J[C] = \|\Phi[C] - f\|_F^2$, comparing the computed output to the measured value, f, in the norm of the output space, F, and seeks to minimize J over K. OLS methods are very general and can be efficiently programmed for computer implementation. Typically there are problems with lack of uniqueness, convergence to false minima and instability under parameter mesh refinement, although a skilful user may be able to incorporate *a priori* information about the solution into the parametric description of the unknown coefficient in order to lessen some of these difficulties [1, 9]. Since the connection between the inputs and outputs is expressed only indirectly through the solver, general information about an input-to-output mapping is not readily available by OLS methods.

An alternative to coefficient identification by output least squares is the so-called equation error method [3, 6, 7, 11, 12]. Here the measured overspecification is used as input to the differential equation in the direct problem which is then viewed as an equation for the unknown coefficient. This equation expresses a direct relationship between the unknown coefficient values and the measured data values. Since the relationship is frequently quite complicated, it is not easy to discern from it properties of an input-to-output mapping. Equation error methods are quite problem dependent and produce varying degrees of success.

The method described in this paper is based on an integral equation relating changes in the unknown coefficient to corresponding changes in the measured output. The integral equation is derived by exploiting a problem which is adjoint to the direct problem, an idea close to the techniques often used to estimate sensitivity in the OLS approach [8, 9]. However, this integral equation provides information about the input/output mapping itself rather than the error functional. It is then possible to prove that the input-to-output map is continuous, monotone and injective. Moreover, it is shown that when the input/output map is restricted to a (finite-dimensional) space of polygonal coefficients, it is explicitly invertible. This observation provides the basis for a method for numerically approximating the unknown coefficient. It is shown that a unique polygonal approximation to the unknown coefficient is obtained by solving a triangular system of linear algebraic equations. Error estimates show that the accuracy of the approximation is limited by the precision of the data measurements so that there is an optimal attainable accuracy but exact determination of the coefficient is never possible.

The results of a few numerical experiments are provided here to illustrate the working of the method. A more extensive presentation of numerical experiments will be included in a later publication.

2. Analysis of the direct and inverse problems

Consider the following IBVP for a quasilinear conduction/diffusion equation on the domain $Q_T = \{0 < x < 1, 0 < t < T\}$,

$$\begin{aligned} &\partial_t u(x,t) = \partial_x(D(u)\partial_x u(x,t)) = \partial_{xx} B(u(x,t)) \qquad \text{on} \quad Q_T, \\ &u(x,0) = f(0) \qquad 0 < x < 1, \\ &u(0,t) = f(t) \qquad \partial_x u(1,t) = 0 \qquad 0 < t < T. \end{aligned} \tag{2.1}$$

Here

$$B(u) = \int_{f(0)}^{u} D(s)\,\mathrm{d}s,$$

and we suppose

$$f \in C^1[0,T] \qquad \text{and} \qquad f'(t) > 0 \qquad \text{for} \quad t > 0. \tag{2.2}$$

For f satisfying (2.2), we let $J = [f(0), f(T)]$, and then suppose that for positive constants, $D_* \leqslant D^*$ and K,

$$\begin{aligned} &D_* \leqslant D(u) \leqslant D^* && \text{for} \quad u \in J, && \text{(i)} \\ &|D(\mu_2) - D(\mu_1)| \leqslant K|\mu_2 - \mu_1| && \forall \mu_1, \mu_2 \in J. && \text{(ii)} \end{aligned} \tag{2.3}$$

Note that any polygonal function (i.e., a continuous and piecewise linear function) satisfies (2.3i) and that any function satisfying both conditions of (2.3) is bounded away from zero and has at most finitely many zeros on J.

Given f satisfying (2.2) and $D(u)$ satisfying (2.3), the so-called direct problem (2.1) has a unique weak solution $u = u(x, t)$ satisfying

$$u \in L^2[0, T : H^1(0, 1) \cap C[0, T : L^2(0, 1)],$$
$$\partial_t u \in L^2[0, T : H^{-1}(0, 1)].$$

Here, we consider the inverse problem in which the coefficient $D = D(u)$ is to be identified from the measured output data. There are a variety of output measurements that are experimentally feasible in any given physical setting; we are going to base our identification on one or the other of the following observations at the boundary:

$$g(t) = -D(u(0, t))\partial_x u(0, t) \qquad \text{or} \qquad h(t) = u(1, t) \qquad 0 < t < T.$$

If we denote the class of uniformly positive, Lipschitz coefficients D satisfying (2.3) by $W(J)$, then for a fixed f satisfying (2.2), we can define mappings

$$\Phi \text{ and } \Psi\colon W(J) \longrightarrow L^2[0, T],$$
$$\Phi[f, D] = g,$$
$$\Psi[f, D] = h,$$

which assign to a coefficient D from $W(J)$, the flux data, g, or the function value data, h, obtained by solving the direct problem (2.1) with inputs f and D. Then solving the inverse problem will amount to inverting these mappings.

We begin with a result about the IBVP (2.1).

Lemma 2.1. *Suppose f and D satisfy (2.2) and (2.3) and let $u = u(x, t)$ denote the corresponding solution of (2.1). Then*

(a) for each $t \in (0, T)$, $f(0) \leqslant u(x, t) \leqslant f(t)$, $0 \leqslant x \leqslant 1$,
(b) $\partial_x u(x, t) < 0$ a.e. on Q_T.

Proof. It follows from (2.1) that,

$$\begin{aligned} &\partial_t[f(t) - u(x, t)] - \partial_{xx}[B(f(t)) - B(u(x, t))] = f'(t) && \text{on} \quad Q_T, \\ &f(0) - u(x, 0) = 0 && 0 < x < 1, \\ &f(t) - u(0, t) = 0 && 0 < t < T, \\ &\partial_x[f(t) - u(1, t)] = 0 && 0 < t < T. \end{aligned}$$

Then we multiply the equation by an arbitrary test function, $\psi(x, t)$, and integrate by parts,

$$\begin{aligned} -\iint_{Q_T} &[(f - u)\partial_t \psi + (B(f) - B(u))\partial_{xx}\psi]\,\mathrm{d}x\,\mathrm{d}t + \int_0^1 (f - u)\psi|_{t=0}^{t=T}\,\mathrm{d}x \\ &- \int_0^T [\psi\partial_x(B(f) - B(u)) - (B(f) - B(u))\partial_x\psi]_{x=0}^{x=1}\,\mathrm{d}t \\ &= \iint_{Q_T} f'(t)\psi\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Note that

$$B(f(t)) - B(u(x,t)) = k(x,t)(f-u),$$

where we define $k(x,t) = D(\mu(x,t))$ for $\mu(x,t)$ between $f(t)$ and $u(x,t)$.

Next we require $\psi(x,t)$ to solve the adjoint problem,

$$\partial_t\psi(x,t) + k(x,t)\partial_{xx}\psi(x,t) = F(x,t) \qquad \text{in} \quad Q_T,$$
$$\psi(x,T) = 0 \qquad 0 < x < 1,$$
$$\psi(0,t) = 0 \qquad \partial_x\psi(1,t) = 0 \qquad 0 < t < T,$$

for a smooth function, $F(x,t)$. Then the integral expression above reduces to

$$-\int\int_{Q_T}(f-u)F(x,t)\,\mathrm{d}x\,\mathrm{d}t = \int\int_{Q_T} f'(t)\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t. \tag{2.4}$$

The smoothness of $k(x,t)$ and $F(x,t)$ implies that the strong maximum principle can be applied to the adjoint problem to conclude that if the function $F(x,t)$ is positive in Q_T, then $\psi(x,t) < 0$ in Q_T. Since f satisfies (2.2), it follows that for every function $F(x,t)$, which is positive in Q_T, the right side of (2.4) is negative, which is to say, for every $F(x,t)$, smooth and positive in Q_T,

$$\int\int_{Q_T}(f-u)F(x,t)\,\mathrm{d}x\,\mathrm{d}t > 0.$$

But this is just the assertion that $f(t) - u(x,t)$ is positive in the sense of distributions on Q_T. Given the smoothness of the solution $u(x,t)$ this means $f(t) > u(x,t)$ almost everywhere on Q_T. Applying the same reasoning to $u(x,t) - f(0)$, we arrive at the expression

$$-\int\int_{Q_T}(u(x,t) - f(0))F(x,t)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T B(f(t))\partial_x\psi(0,t)\,\mathrm{d}t.$$

where we again use that $\psi(x,t) < 0$ in Q_T if the function $F(x,t)$ is positive in Q_T. Now this fact, together with the adjoint boundary conditions, implies that $\partial_x\psi(0,t) < 0$, for $0 < t < T$. Then the conclusion follows as before. This completes the proof of (*a*).

To prove (*b*), multiply both sides of (2.1) by $\partial_x\phi(x,t)$ for an arbitrary test function $\phi(x,t)$ and use integration by parts to arrive at

$$0 = \int\int_{Q_T}\partial_x u[\partial_t\phi + D(u)\partial_{xx}\phi]\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \phi\partial_t u|_{x=0}^{x=1}\mathrm{d}t$$
$$- \int_0^1 \phi\partial_x u|_{t=0}^{t=T}\mathrm{d}x - \int_0^T \partial_x\phi\partial_x B(u)|_{x=0}^{x=1}\,\mathrm{d}t.$$

Now require that $\phi(x,t)$ satisfies the adjoint problem

$$\partial_t\phi(x,t) + D(u(x,t))\partial_{xx}\phi(x,t) = F(x,t) \qquad \text{in} \quad Q_T,$$
$$\phi(x,T) = 0 \qquad 0 < x < 1,$$
$$\partial_x\phi(0,t) = 0 \qquad \phi(1,t) = 0 \qquad 0 < t < T.$$

Then the preceding integral expression reduces to

$$\int\int_{Q_T}\partial_x u(x,t)F(x,t)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \phi(0,t)f'(t)\,\mathrm{d}t.$$

The maximum principle can be applied to the adjoint problem to conclude that $\phi(x,t) < 0$ in Q_T if the continuous function $F(x,t)$ is positive in Q_T. In particular, $\phi(0,t) < 0$ for $0 < t < T$ and since f satisfies (2.2), it follows that for every function $F(x,t)$ which is positive in Q_T the right side of the expression is negative. Then it follows as in the proof of part (a) that $\partial_x u(x,t) < 0$ almost everywhere in Q_T. □

The results of this lemma are crucial to the proof of

Lemma 2.2. *Suppose f satisfies (2.2) and D_1, D_2 both satisfy (2.3). Then if $D_1(u) > D_2(u)$ for $u \in J = [f(0), f(T)]$ it follows that*

$$(a)\ \Phi[f, D_1](t) > \Phi[f, D_2](t) \qquad 0 < t < T,$$
$$(b)\ \Psi[f, D_1](t) < \Psi[f, D_2](t) \qquad 0 < t < T.$$

Proof. For $w \in J$, let $B_j'(w) = D_j(w)$, $j = 1, 2$, and let u_1, u_2 denote the solutions for the direct problem with coefficients D_1, D_2, respectively. Then

$$\partial_t(u_1 - u_2) - \partial_{xx}(B_1(u_1) - B_1(u_2)) = \partial_{xx}(B_1(u_2) - B_2(u_2)),$$

and, for an arbitrary test function $\phi = \phi(x,t)$, and arbitrary τ, $0 < \tau \leqslant T$,

$$\int_0^\tau \int_0^1 [\partial_t(u_1 - u_2) - \partial_{xx}(B_1(u_1) - B_1(u_2))]\,\phi\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_0^\tau \int_0^1 \phi\,\partial_{xx}(B_1(u_2) - B_2(u_2))\,\mathrm{d}x\,\mathrm{d}t.$$

Apply integration by parts on the left side of this equation,

$$\int_0^\tau \int_0^1 [\partial_t(u_1 - u_2) - \partial_{xx}(B_1(u_1) - B_1(u_2))]\,\phi\,\mathrm{d}x\,\mathrm{d}t$$
$$= -\int_0^\tau \int_0^1 (u_1 - u_2)\{\partial_t\phi + D_1(\mu(x,t))\partial_{xx}\phi\}\,\mathrm{d}x\,\mathrm{d}t + \int_0^1 (u_1 - u_2)\phi|_{t=0}^{t=\tau}\,\mathrm{d}x$$
$$- \int_0^\tau [\phi\partial_x(B_1(u_1) - B_1(u_2)) - \partial_x\phi(B_1(u_1) - B_1(u_2))]_{x=0}^{x=1}\,\mathrm{d}t,$$

and on the right side as well,

$$\int_0^\tau \int_0^1 \phi\{\partial_{xx}(B_1(u_2) - B_2(u_2))\}\,\mathrm{d}x\,\mathrm{d}t = \int_0^\tau [\phi\partial_x(B_1(u_2) - B_2(u_2))]_{x=0}^{x=1}\,\mathrm{d}t$$
$$- \int_0^\tau \int_0^1 (D_1(u_2) - D_2(u_2))\partial_x\phi\partial_x u_2\,\mathrm{d}x\,\mathrm{d}t,$$

where for all $(x,t) \in Q_\tau$, $\mu(x,t)$ lies between $u_1(x,t)$ and $u_2(x,t)$ such that for $(x,t) \in Q_\tau$

$$B_1(u_1(x,t)) - B_1(u_2(x,t)) = D_1(\mu(x,t))[u_1(x,t) - u_2(x,t)].$$

Then we obtain the following integral expression

$$-\int_0^\tau \int_0^1 (u_1 - u_2)\{\partial_t\phi + D_1(\mu(x,t))\partial_{xx}\phi\}\,\mathrm{d}x\,\mathrm{d}t + \int_0^1 (u_1 - u_2)\phi|_{t=0}^{t=\tau}\,\mathrm{d}x$$
$$- \int_0^\tau [\phi\partial_x(B_1(u_1) - B_2(u_2)) - \partial_x\phi(B_1(u_1) - B_1(u_2))]_{x=0}^{x=1}\,\mathrm{d}t$$
$$= -\int_0^\tau \int_0^1 (D_1(u_2) - D_2(u_2))\partial_x\phi\partial_x u_2\,\mathrm{d}x\,\mathrm{d}t.$$

The boundary and initial conditions of the direct problem cause this expression to reduce to

$$
\begin{aligned}
-\int_0^\tau \int_0^1 (u_1 - u_2)\{\partial_t\phi + D_1(\mu(x,t))\partial_{xx}\phi\}\,\mathrm{d}x\,\mathrm{d}t + \int_0^1 (u_1-u_2)(x,\tau)\phi(x,\tau)\,\mathrm{d}x \\
+\int_0^\tau \phi(0,t)\partial_x(B_1(u_1) - B_2(u_2))\,\mathrm{d}t - \int_0^\tau \partial_x\phi(1,t)[B_1(u_1) - B_1(u_2)]\,\mathrm{d}t \\
= -\int_0^\tau \int_0^1 (D_1(u_2) - D_2(u_2))\partial_x\phi\partial_x u_2\,\mathrm{d}x\,\mathrm{d}t. \qquad (2.5)
\end{aligned}
$$

Now require the arbitrary function $\phi(x,t)$ to solve the so-called g-adjoint problem,

$$
\begin{aligned}
&\partial_t\phi + D_1(\mu(x,t))\partial_{xx}\phi = 0 && \text{in} \quad Q_\tau, \\
&\phi(x,\tau) = 0 && 0 < x < 1, \\
&\phi(0,t) = \theta(t) && 0 < t < \tau, \\
&\partial_x\phi(1,t) = 0 && 0 < t < \tau,
\end{aligned} \qquad (2.6)
$$

where $\theta(t) = F(\tau - t)$ and F is any function satisfying (2.2). Then (2.5) reduces to

$$
\int_0^\tau \theta(t)[g_1(t) - g_2(t)]\,\mathrm{d}t = \int_0^\tau \int_0^1 (D_1(u_2) - D_2(u_2))\partial_x\phi\partial_x u_2\,\mathrm{d}x\,\mathrm{d}t. \qquad (2.7)
$$

An argument similar to that used in the proof of the previous lemma, applied to (2.6), shows that the assumption on the adjoint input, θ, implies $\partial_x\phi(x,t) < 0$ on Q_τ. Since $\partial_x u_2 < 0$ on Q_T and $D_1(u_2) > D_2(u_2)$ it follows that the right side of the last expression is positive. Since (2.7) holds for all $\theta(t) = F(\tau - t)$, such that F satisfies (2.2), it follows that

$$
g_1(t) - g_2(t) > 0 \qquad \text{for} \quad 0 < t < T,
$$

i.e.,

$$
g_1(t) = \Phi[f, D_1](t) > \Phi[f, D_2](t) = g_2(t).
$$

To see that this is true, note first that if $D_1(u) > D_2(u)$ for $u \in J$, then existence of an interval $(0, t_1)$ with $g_1(t) < g_2(t)$ for $0 < t < t_1$ is precluded by (2.7) simply by choosing $\tau = t_1$. Suppose then that there exists $t_2 > t_1 > 0$ such that $g_1(t) \geqslant g_2(t)$ for $0 < t \leqslant t_1$ and $g_1(t) < g_2(t)$ for $t_1 < t < t_2$. Then choosing $\tau = t_2$ in (2.7) implies that for any admissible $\theta(t)$,

$$
\begin{aligned}
\int_{t_1}^{t_2} \theta(t)[g_1(t) - g_2(t)]\,\mathrm{d}t = \int_{t_1}^{t_2} \int_0^1 (D_1(u_2) - D_2(u_2))\partial_x\phi\partial_x u_2\,\mathrm{d}x\,\mathrm{d}t \\
+\int_0^{t_1} \int_0^1 (D_1(u_2) - D_2(u_2))\partial_x\phi\partial_x u_2\,\mathrm{d}x\,\mathrm{d}t - \int_0^{t_1} \theta(t)[g_1(t) - g_2(t)]\,\mathrm{d}t.
\end{aligned}
$$

By applying equality (2.7) with $\tau = t_1$, the last two terms of the previous equation vanish. By assumption, the right side of the resulting expression is strictly positive, while a suitable choice of $\theta(t)$ makes the left side non-negative. This contradicts (2.7).

Suppose now that we choose ϕ in (2.5) to solve a problem different from (2.6). This problem will be called the h-adjoint problem,

$$
\begin{aligned}
&\partial_t\phi + D_1(\mu(x,t))\partial_{xx}\phi = 0 && \text{in} \quad Q_\tau, \\
&\phi(x,\tau) = 0 && 0 < x < 1, \\
&\phi(0,t) = 0 && 0 < t < \tau \\
&D(\mu(1,t))\partial_x\phi(1,t) = \beta(t) && 0 < t < \tau.
\end{aligned} \qquad (2.8)
$$

Here, choose $\beta(t) = F(\tau - t)$ where F is any function satisfying (2.2). Then (2.5) reduces to

$$\int_0^\tau D(\mu(1,t))\partial_x\phi(1,t)(u_1(1,t) - u_2(1,t))\,\mathrm{d}t = \int_0^\tau \int_0^1 (D_1(u_2) - D_2(u_2))\partial_x\phi\partial_x u_2\,\mathrm{d}x\,\mathrm{d}t$$

or

$$\int_0^\tau \beta(t)[h_1(t) - h_2(t)]\,\mathrm{d}t = \int_0^\tau \int_0^1 (D_1(u_2) - D_2(u_2))\partial_x\phi\partial_x u_2\,\mathrm{d}x\,\mathrm{d}t. \tag{2.9}$$

In this case, the hypotheses on $\beta(t)$ imply that $\partial_x\phi(x,t) > 0$ on Q_τ and since $\partial_x u_2 < 0$ and $D_1(u_2) > D_2(u_2)\ \forall u_2 \in J$, it follows that the right side of (2.9) is negative. Since this holds with $\beta(t) = F(\tau - t)$ for any F satisfying (2.2), it follows that

$$\Psi[f, D_1](t) = u_1(1,t) < u_2(1,t) = \Psi[f, D_2](t) \qquad \text{for} \quad 0 < t < \tau.$$

Finishing the argument as in the previous case, we see that this holds for $\tau \leqslant T$. □

The conclusions of lemma 2.2 assert that input-to-output mappings Φ and Ψ are monotone mappings. More precisely, the mapping Φ is *isotone* while the mapping Ψ is an *antitone* mapping.

Now suppose $D_1(u_1)$ and $D_2(u_1)$ are any two coefficients, both satisfying (2.3). Let $u_1(x,t)$, $u_2(x,t)$ denote the solutions of (2.1) when the coefficient is, respectively, $D_1(u)$ and $D_2(u)$, and for $i = 1, 2$, let

$$g_i(t) = \Phi[f, D_i] \qquad \text{and}$$
$$h_i(t) = \Psi[f, D_i] \qquad 0 < t < T.$$

Now choose the data in the adjoint problems (2.6) and (2.8) as

$$\phi(0,t) = \theta(t) = \frac{g_1(t) - g_2(t)}{\|g_1 - g_2\|_{L^2[0,T]}}, \qquad \text{in (2.6)}$$

and

$$D_1(\mu(1,t))\partial_x\psi(1,t) = \beta(t) = \frac{h_1(t) - h_2(t)}{\|h_1 - h_2\|_{L^2[0,T]}} \qquad \text{in (2.8)}.$$

It follows at once from (2.7) that

$$\begin{aligned}\|g_1 - g_2\|_{L^2[0,T]} &\leqslant \left|\int_0^T \int_0^1 (D_1(u_2) - D_2(u_2))\partial_x u_2\partial_x\phi\,\mathrm{d}x\,\mathrm{d}t\right| \\ &\leqslant C\|D_1 - D_2\|_\infty,\end{aligned}$$

and from (2.9) that

$$\begin{aligned}\|h_1 - h_2\|_{L^2[0,T]} &\leqslant \left|\int_0^T \int_0^1 (D_1(u_2) - D_2(u_2))\partial_x u_2\partial_x\psi\,\mathrm{d}x\,\mathrm{d}t\right| \\ &\leqslant C\|D_1 - D_2\|_\infty.\end{aligned}$$

Evidently, this is just the assertion that Φ and Ψ are continuous as a function of D from $W(J)$ into $L^2[0,T]$; i.e.,

$$\|g_1 - g_2\|_{L^2[0,T]} = \|\Phi(f, D_1) - \Phi(f, D_2)\|_{L^2[0,T]} \leqslant C\|D_1 - D_2\|_\infty,$$
$$\|h_1 - h_2\|_{L^2[0,T]} = \|\Psi(f, D_1) - \Psi(f, D_2)\|_{L^2[0,T]} \leqslant C\|D_1 - D_2\|_\infty.$$

Having shown that Φ and Ψ are continuous and strictly monotone, one is encouraged to believe that this inverse problem is not so badly ill-posed and that Φ and Ψ might be continuously invertible. Such a strong result seems to be unlikely without a simple ordering on the domain and range of these maps but it is at least true that the input/output maps Φ and Ψ are injective as the following lemma shows.

Lemma 2.3. *For a fixed f satisfying (2.2) and coefficients* $D_1, D_2 \in W(J)$ *let* $g_k(t) = \Phi[f, D_k]$ *and* $h_k(t) = \Psi[f, D_k t]$, *for* $k = 1, 2$.

Then

(*a*) $\Phi[f, D_1] = \Phi[f, D_2], 0 < t < T$ *implies* $D_1(u) = D_2(u)$ *for* $u \in J$.

(*b*) $\Psi[f, D_1] = \Psi[f, D_2], 0 < t < T$ *implies* $D_1(u) = D_2(u)$ *for* $u \in J$.

Proof. Suppose first that $D_1(f(0)) = D_2(f(0))$. Now, since D_1 and D_2 both satisfy (2.3), their difference satisfies (2.3) and if these functions are not identical on J then there exists a positive time $t_1, 0 < t_1 \leqslant T$, where the difference, $D_1(f(t)) - D_2(f(t))$ is of one sign on $[0, t_1]$. Then lemma 2.1(a) implies $D_1(u_2(x,t)) - D_2(u_2(x,t))$ is of one sign on $(0,1) \times (0, t_1)$. Using the identity (2.7), we have

$$\int_0^{t_1} \int_0^1 (D_1(u_2(x,t)) - D_2(u_2(x,t)))\partial_x u_2 \partial_x \phi \,\mathrm{d}x\,\mathrm{d}t = \int_0^{t_1} (g_1(t) - g_2(t))\theta(t)\,\mathrm{d}t,$$

where ϕ solves (2.6) with $\tau = t_1$. Then the hypotheses imply that the right side of this equation vanishes, i.e.,

$$\int_0^{t_1} \int_0^1 (D_1(u_2) - D_2(u_2))\partial_x u_2 \partial_x \phi \,\mathrm{d}x\,\mathrm{d}t = 0,$$

and this holds independently of the data $\theta(t)$ chosen as input to the adjoint problem. It is clearly possible to choose $\theta(t)$ so that $\partial_x \phi < 0$ on $(0,1) \times (0, t_1)$ and in view of lemma (b) it is also the case that $\partial_x u_2 < 0$ on $(0,1) \times (0, t_1)$. Then the vanishing integral above has an integrand which is of one sign over the domain of integration and vanishes on no positive measure subset of the domain. This contradiction is in opposition to the assumption that D_1 and D_2 are not identical.

If we suppose $D_1(f(0)) \neq D_2(f(0))$ then it follows that either there is a smallest time $t_1, 0 < t_1 < T$, where the difference $D_1(f(t)) - D_2(f(t))$ is zero, or else $t_1 = T$ and the difference is of one sign on $[0, T]$. In either case, it is evident that $D_1(f(t)) - D_2(f(t))$ is of one sign on $[0, t_1], 0 < t_1 \leqslant T$, and the argument can be completed as before. A similar argument, using the identity in (2.9), establishes conclusion (b). □

Formally, we can write

$$\begin{aligned}(\Phi[f, D_1] - \Phi[f, D_2], \theta)_{L^2} &\stackrel{\text{def}}{=} (\delta\Phi[D_1, D_2]\Delta D, \theta)_{L^2} \\ &= \langle \Delta D, {}^t\delta\Phi[D_1, D_2]\theta\rangle_{W(J)\times W(J)^*}.\end{aligned}$$

In view of (2.7),

$$\begin{aligned}(\Phi[f, D_1] - \Phi[f, D_2], \theta)_{L^2} &= \int_0^T (g_1(t) - g_2(t))\theta(t)\,\mathrm{d}t \\ &= \int_0^T \int_0^1 (D_1(u_2) - D_2(u_2))\partial_x u_2 \partial_x \phi \,\mathrm{d}x\,\mathrm{d}t, \\ &= \langle \Delta D, {}^t\delta\Phi[D_1, D_2]\theta\rangle_{W(J)\times W(J)^*}.\end{aligned}$$

Similarly,

$$\begin{aligned}(\Psi[f, D_1] - \Psi[f, D_2], \beta)_{L^2} &\stackrel{\text{def}}{=} (\delta\Psi[D_1, D_2]\Delta D, \beta)_{L^2} \\ &= \langle \Delta D, {}^t\delta\Psi[D_1, D_2]\beta\rangle_{W(J)\times W(J)^*},\end{aligned}$$

and, referring to (2.9),

$$(\Psi[f, D_1] - \Psi[f, D_2], \beta)_{L^2} = \int_0^T (h_1(t) - h_2(t))\beta(t)\,\mathrm{d}t$$
$$= \int_0^T \int_0^1 (D_1(u_2) - D_2(u_2))\partial_x u_2 \partial_x \psi \,\mathrm{d}x\,\mathrm{d}t$$
$$= \langle \Delta D, {}^t\delta\Psi[D_1, D_2]\beta\rangle_{W(J)\times W(J)^*}.$$

Evidently, (2.7), (2.9) provide realizations for ${}^t\delta\Phi[D_1, D_2]$ and ${}^t\delta\Psi[D_1, D_2]$, the Gateaux derivatives with respect to D of the mappings Φ and Ψ. It will be shown in the next section that ${}^t\delta\Phi[D_1, D_2]$ and ${}^t\delta\Psi[D_1, D_2]$ are invertible in an approximate sense. More precisely we will devise a restriction of the coefficient to data maps that induces a mapping from $\mathbb{R}$ into $\mathbb{R}$. The restriction inherits the strict monotonicity and continuity from the coefficient to data map hence the restriction defines a homeomorphism from its domain onto its range. Inversion of this mapping leads to an approximate inverse for the coefficient to data map.

3. The approximate solution of the inverse problem

We consider the inverse problem in which the coefficient $D = D(u)$ is to be identified from data which are assumed to be recorded at fixed nodes $0 = t_0 < t_1 < \cdots < t_N = T$ in the interval $[0, T]$:

$$\text{data}(f, g)\begin{cases} f(t_k) = \mu_k \\ g(t_k) = -D(\mu_k)\partial_x u_1(0, t_k) = \gamma_k \qquad k = 0, 1, \ldots, N. \end{cases}$$

We are also interested in the identification of $D = D(u_1)$ based on the alternative data,

$$\text{data}(f, h)\begin{cases} f(t_k) = \mu_k \\ h(t_k) = u_1(1, t_k) = \eta_k \qquad k = 0, 1, \ldots, N. \end{cases}$$

More precisely, we are going use one or the other of these data sets to construct a polygonal (i.e., piecewise linear and continuous) approximation to the unknown coefficient $D(u)$. The data set, $f_k = f(t_k), k = 0, 1, \ldots, N$, is assumed to be given at fixed nodes which define a partition, $0 = t_0 < t_1 < \cdots < t_N = T$, of the interval $I = [0, T]$. This partition of I will be called the 'inner mesh'. We then define an associated (but coarser) partition of $J = [f(0), f(T)]$, the domain of the coefficient D. This partition will be called the 'outer mesh' and is given by $f(0) = \mu_0 < \mu_1 < \cdots < \mu_M = f(T)$, i.e., $\mu_0 = f_0$, and $\mu_M = f_N$ and for each $j = 1, \ldots, M < N$, we have $\mu_j = f_k$ for some $k \geqslant j$.

It is necessary for the outer mesh to be coarser than the inner mesh since on each subinterval in the outer mesh, we will need to compute interior values of the solution $u(x, t)$, for the direct problem in order to be able to evaluate the integrals which appear in the identities used in the identification. Between two outer mesh knots $\mu_j = f(t_k)$ and μ_{j+1}, several inner mesh knots must occur and this fact prevents the outer mesh from being made arbitrarily fine in order to improve the accuracy of the identification.

We can now consider a family of polygonal functions, $\hat{D}$, associated with the partition of J. Each member of the family is characterized by its values at the nodes μ_k, i.e. for $d_k = \hat{D}(\mu_k)$. More precisely, we define

$$\hat{D}(u) = \sum_{k=1}^{M} [d_{k-1}\rho_{k-1}(u) + d_k\lambda_k(u)] \tag{3.1}$$

where

$$\rho_k(u) = \begin{cases} \dfrac{\mu_{k+1} - u}{\mu_{k+1} - \mu_k} & \text{if} \quad \mu_k \leqslant u \leqslant \mu_{k+1} \\ 0 & \text{otherwise} \end{cases} \qquad 1 \leqslant k \leqslant M - 1,$$

$$\lambda_k(u) = \begin{cases} \dfrac{u - \mu_{k-1}}{\mu_k - \mu_{k-1}} & \text{if} \quad \mu_{k-1} \leqslant u \leqslant \mu_k \\ 0 & \text{otherwise} \end{cases} \qquad 1 \leqslant k \leqslant M.$$

Equivalently, we could write for $1 \leqslant k \leqslant M$,

$$\hat{D}(u) = d_{k-1}\rho_{k-1}(u) + d_k\lambda_k(u) \qquad \text{for} \quad \mu_{k-1} \leqslant u \leqslant \mu_k. \tag{3.2}$$

We will introduce the following notation:

- $\hat{D}(u) = P_M[d_0, d_1, \ldots, d_M]$ denotes the polygonal coefficient given by (3.1) based on nodal values $[d_0, d_1, \ldots, d_M]$.
- $u(x, t; D, f)$ denotes the solution of the direct problem (2.1) with coefficient D and data, f.
- $\phi(x, t, D, \theta)$ denotes the solution of the adjoint problem (2.6) with coefficient $D(x, t) \stackrel{\text{def}}{=} D(\mu(x, t))$ and data, $\theta(t)$.
- $\psi(x, t, D, \beta)$ denotes the solution of the adjoint problem (2.8) with coefficient $D(x, t) \stackrel{\text{def}}{=} D(\mu(x, t))$ and data, $\beta(t)$.

For a given $f(t)$ satisfying (2.2), an unknown coefficient $D = D(u)$ satisfying (2.3) and measured flux data $g(t) = \Phi[f, D]$, we assume there is a fixed outer partition, $\Pi = \{0 = \mu_0 < \mu_1 < \cdots < \mu_M = f(T)\}$ of J. Then we will define a polygonal coefficient approximation to D by the following recursive algorithm based on (f, g) − data, $\{f(t), g(t)\}$:

- d_0 is assumed to be given
- for $k = 1, 2, \ldots, d_k$ is determined from $d_0, d_1, \ldots, d_{k-1}$ by
- $$(d_k - d_{k-1})\int_{T_{k-1}}^{T_k} \lambda_k(u_2)\partial_x u_2 \partial_x \phi \, \mathrm{d}x \, \mathrm{d}t = -\int_{T_{k-1}}^{T_k} (g(t) - g_2(t))\theta(t) \, \mathrm{d}t, \tag{3.3}$$

where

$$\begin{aligned} &D_1(u) = P_M[d_0, d_1, \ldots, d_{k-1}, d_k], \\ &D_2(u) = P_M[d_0, d_1, \ldots, d_{k-1}, d_{k-1}], \\ &u_2(x, t) = u(x, t; D_2, f), \\ &g_2(t) = -D_2(f(t))\partial_x u_2(0, t) \qquad 0 \leqslant x \leqslant 1, 0 \leqslant t \leqslant T_k, \\ &\phi(x, t) = \phi(x, t; D_1, f(T - t)) \qquad \text{for} \quad 0 \leqslant x \leqslant 1, 0 \leqslant t \leqslant T_k. \end{aligned}$$

The approximation of $D(u)$ based on (f, h) − data, $\{f(t), h(t)\}$, is analogous. We can show then

Lemma 3.1. *For $f(t)$ satisfying (2.2), for coefficient D satisfying (2.3) and for a fixed partition, $\Pi = 0 = \mu_0 < \mu_1 < \cdots < \mu_M = f(T)$ of J, let the nodal values $[d_0, d_1, \ldots, d_M]$ be determined by the algorithm (3.3). Then for $k = 1, 2, \ldots, M$,*

$$|D(\mu_k) - d_k| \leqslant C|\mu_k - \mu_{k-1}|. \tag{3.4}$$

Proof. We are going to assume that the initial nodal value, $D(\mu_0) = D(f(0)) = d_0$, is known and that the remaining values $d_1, \ldots, d_M$ are determined by the algorithm (3.3). Consider first the value d_1. If we apply the identity (2.7) with $\tau = T_1$, and,

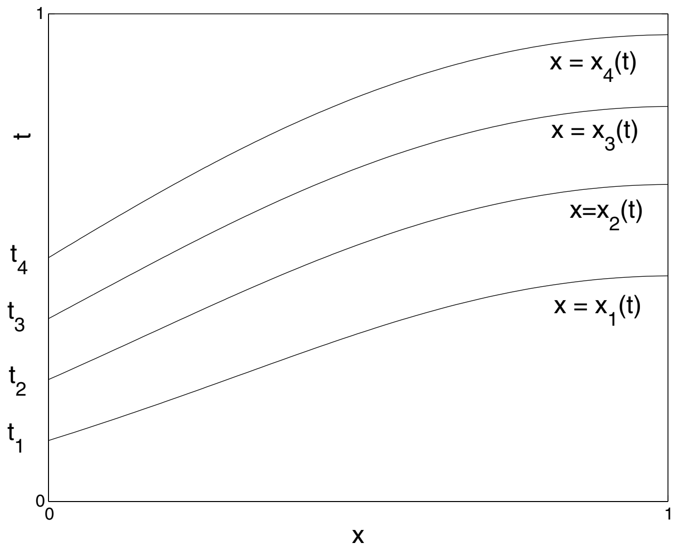

Figure 1. Isocurves.

- on $J_1 = [\mu_0, \mu_1]$, $D_1(u) = P_M[d_0, d_1]$, and $D_2(u) = P_M[d_0, d_0]$,
- on $Q_1 = (0, 1) \times (0, T_1) u_1(x, t) = u(x, t; D_1, f)$, and $u_2(x, t) = u(x, t; D_2, f)$,

then we have

$$\int_0^{T_1} \int_0^1 (D_1(u_2) - D_2(u_2)) \partial_x u_2 \partial_x \phi \, \mathrm{d}x \, \mathrm{d}t = \int_0^{T_1} (g(t) - g_2(t)) \theta(t) \, \mathrm{d}t.$$

Here $g(t)$ is the measured flux data and $g_2(t)$ is the output generated by solving (2.1) with the coefficient $D(u) = D_2(u)$, i.e., $g_2 = \Phi[f, D_2]$. The functions $\theta(t)$ and $\phi(x, t)$ denote the data and solution respectively for the g-adjoint problem. Since the function $f(t)$ in the direct problem satisfies (2.2), it follows from lemma 2.1(a) that u_2 satisfies

$$f(0) = \mu_0 \leqslant u_2(x, t) \leqslant \mu_1 = f(T_1) \qquad \text{for} \quad (x, t) \in (0, 1) \times (0, T_1).$$

Then according to (3.2) for $u \in J_1 = \mu_0 \leqslant u \leqslant \mu_1$,

$$D_1(u) = d_0 \rho_0(u) + d_1 \lambda_1(u), \qquad D_2(u) = d_0 \rho_0(u) + d_0 \lambda_1(u) = d_0,$$

and so

$$D_1(u_2) - D_2(u_2) = (d_1 - d_0) \lambda_1(u_2).$$

Note that for each nodal value, $\mu_k, 0 \leqslant k \leqslant M$, we have $u_2(x_k(t), t) = \mu_k$ along some curve $x = x_k(t)$, with $x_k(0) = \mu_k$ and $x_k(\tau_k) = 1$ for some $\tau_k > \tau_{k-1} > \cdots > \tau_1 > 0$. Examples of such curves are shown in figure 1.

Then we have $u_2(x(t), t) = \mu_0$ along a curve $x = x_0(t)$, with $x_0(0) = 0$ and $x_0(\tau_1) = 1$ for some $\tau_1 > 0$. We suppose further that T_1 is sufficiently small that $0 < x_0(T_1) < 1$. Then

$$\lambda_1(u_2(x, t)) = \begin{Bmatrix} \dfrac{\mu_0 - u_2(x, t)}{\mu_0 - \mu_1} & \text{if} \quad 0 \leqslant x \leqslant x_0(t), 0 \leqslant t \leqslant T_1 \\ 0 & \text{if} \quad x > x_0(t), 0 \leqslant t \leqslant T_1 \end{Bmatrix}$$

and the integral identity reduces to,

$$(d_1 - d_0) \int_0^{T_1} \int_0^{x_0(t)} \lambda_1(u_2) \partial_x u_2 \partial_x \phi \, \mathrm{d}x \, \mathrm{d}t = \int_0^{T_1} (g(t) - g_2(t)) \theta(t) \, \mathrm{d}t,$$

i.e.,

$$d_1 = d_0 + \frac{\int_0^{T_1}(g(t) - g_2(t))\theta(t)\,\mathrm{d}t}{\int_0^{T_1}\int_0^{x_0(t)} \lambda_1(u_2)\partial_x u_2 \partial_x \phi\,\mathrm{d}x\,\mathrm{d}t}.$$

This equation defines the first unknown nodal value d_1. Now we will establish the relationship between d_1 and $D(\mu_1)$. It follows from (2.7) that

$$\begin{aligned}\int_0^{T_1}\int_0^{x_0(t)}(D(u_2) - D_2(u_2))\partial_x u_2 \partial_x \phi\,\mathrm{d}x\,\mathrm{d}t &= \int_0^{T_1}(g(t) - g_2(t))\theta(t)\,\mathrm{d}t \\ &= (d_1 - d_0)\int_0^{T_1}\int_0^{x_0(t)}\lambda_1(u_2)\partial_x u_2 \partial_x \phi\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Let $\hat{D}_M(u)$ denote the polygonal coefficient on the partition Π which satisfies $\hat{D}_M(\mu_k) = D(\mu_k)$ for all k. Note that this coefficient does not, in general, generate the given measured data, $g(t)$, and is not then the polygonal coefficient with nodal values $\{d_k\}$ generated by the algorithm. However, these coefficients are related as follows:

$$\begin{aligned}&\int_0^{T_1}\int_0^{x_0(t)}(D(u_2) - D_2(u_2))\partial_x u_2 \partial_x \phi\,\mathrm{d}x\,\mathrm{d}t \\ &\quad= \int_0^{T_1}\int_0^{x_0(t)}(D(u_2) - \hat{D}_M(u_2))\partial_x u_2 \partial_x \phi\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad+ \int_0^{T_1}\int_0^{x_0(t)}(\hat{D}_M(u_2) - D_2(u_2))\partial_x u_2 \partial_x \phi\,\mathrm{d}x\,\mathrm{d}t \\ &\quad= \int_0^{T_1}\int_0^{x_0(t)}(D(u_2) - \hat{D}_M(u_2))\partial_x u_2 \partial_x \phi\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad+ (D(\mu_1) - d_0)\int_0^{T_1}\int_0^{x_0(t)}\lambda_1(u_2)\partial_x u_2 \partial_x \phi\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

and by combining these two expressions it follows that

$$\begin{aligned}&(d_1 - D(\mu_1))\int_0^{T_1}\int_0^{x_0(t)}\lambda_1(u_2)\partial_x u_2 \partial_x \phi\,\mathrm{d}x\,\mathrm{d}t \\ &\quad= \int_0^{T_1}\int_0^{x_0(t)}(D(u_2) - \hat{D}_M(u_2))\partial_x u_2 \partial_x \phi\,\mathrm{d}x\,\mathrm{d}t \\ &\quad\leqslant \max_{J_1}|D - \hat{D}_M|\left|\int_0^{T_1}\int_0^{x_0(t)}\partial_x u_2 \partial_x \phi\,\mathrm{d}x\,\mathrm{d}t\right|.\end{aligned}$$

Now

$$\max_{J_1}|D - \hat{D}_M| = |D(\mu_*) - \hat{D}_M(\mu_*)| \qquad \text{for some} \quad \mu_* \in J_1.$$

But

$$\begin{aligned}|D(\mu_*) - \hat{D}_M(\mu_*)| &\leqslant |D(\mu_*) - D(\mu_0)| + |D(\mu_0) - \hat{D}_M(\mu_*)| \\ &\leqslant K|\mu_* - \mu_0| + |\hat{D}_M(\mu_0) - \hat{D}_M(\mu_*)|.\end{aligned}$$

In addition,

$$|\hat{D}_M(\mu_0) - \hat{D}_M(\mu_*)| \leqslant K|\mu_* - \mu_0|,$$

and

$$|D(\mu_*) - \hat{D}_M(\mu_*)| \leqslant 2K|\mu_* - \mu_0|.$$

Then

$$|d_1 - D(\mu_1)| \leqslant 2K \frac{\left|\int_0^{T_1}\int_0^{x_0(t)} \partial_x u_2 \partial_x \phi \, \mathrm{d}x \, \mathrm{d}t\right|}{\left|\int_0^{T_1}\int_0^{x_0(t)} \lambda_1(u_2) \partial_x u_2 \partial_x \phi \, \mathrm{d}x \, \mathrm{d}t\right|} |\mu_* - \mu_0|.$$

Since it is clear that for some λ_1^*, $0 < \lambda_1^* < 1$,

$$\int_0^{T_1}\int_0^{x_0(t)} \lambda_1(u_2)\partial_x u_2 \partial_x \phi \, \mathrm{d}x \, \mathrm{d}t = \lambda_1^* \int_0^{T_1}\int_0^{x_0(t)} \partial_x u_2 \partial_x \phi \, \mathrm{d}x \, \mathrm{d}t$$

we find

$$1 \leqslant \frac{\left|\int_0^{T_1}\int_0^{x_0(t)} \partial_x u_2 \partial_x \phi \, \mathrm{d}x \, \mathrm{d}t\right|}{\left|\int_0^{T_1}\int_0^{x_0(t)} \lambda_1(u_2) \partial_x u_2 \partial_x \phi \, \mathrm{d}x \, \mathrm{d}t\right|} \leqslant \frac{1}{\lambda_1^*} < \infty.$$

Then

$$|d_1 - D(\mu_1)| \leqslant \frac{2K}{\lambda_1^*} |\mu_* - \mu_0| \leqslant C_1 |\mu_1 - \mu_0| .$$

This is the result (3.4) for $k = 1$.

In determining the succeeding values d_k, we assume $d_0, d_1, \ldots, d_{k-1}$ are known and we let

- on $[\mu_0, \mu_k]$, $D_1(u) = P_M[d_0, d_1, \ldots, d_{k-1}, d_k]$ and $D_2(u) = P_M[d_0, d_1, \ldots, d_{k-1}, d_{k-1}]$,
- on $Q_k = (0, 1) \times (0, T_k)$, $u_1(x, t) = u(x, t; D_1, f)$ and $u_2(x, t) = u(x, t; D_2, f)$.

Then $D_1(u)$ and $D_2(u)$ are identical on $[\mu_0, \mu_{k-1}]$ and only differ on $J_k = [\mu_{k-1}, \mu_k]$ where we have

$$D_1(u) = d_{k-1}\rho_{k-1}(u) + d_k\lambda_k(u) \qquad \text{for} \quad \mu_{k-1} \leqslant u \leqslant \mu_k,$$
$$D_2(u) = d_{k-1}\rho_{k-1}(u) + d_{k-1}\lambda_k(u) = d_{k-1} \qquad \text{for} \quad \mu_{k-1} \leqslant u \leqslant \mu_k.$$

Then

$$\int_0^{T_k}\int_0^1 (D_1(u_2) - D_2(u_2))\partial_x u_2 \partial_x \phi \, \mathrm{d}x \, \mathrm{d}t = \int_{T_{k-1}}^{T_k}\int_0^1 (D_1(u_2) - D_2(u_2))\partial_x u_2 \partial_x \phi \, \mathrm{d}x \, \mathrm{d}t$$
$$= (d_k - d_{k-1}) \int_{T_{k-1}}^{T_k}\int_0^{x_{k-1}(t)} \lambda_k(u_2)\partial_x u_2 \partial_x \phi \, \mathrm{d}x \, \mathrm{d}t.$$

and we have

$$(d_k - d_{k-1}) \int_{T_{k-1}}^{T_k}\int_0^{x_{k-1}(t)} \lambda_k(u_2)\partial_x u_2 \partial_x \phi \, \mathrm{d}x \, \mathrm{d}t = \int_{T_{k-1}}^{T_k} (g(t) - g_2(t))\theta(t) \, \mathrm{d}t,$$

as prescribed by (3.3). Now we proceed as in the first part of the proof to show that

$$|d_k - D(\mu_k)| \leqslant C|\mu_k - \mu_{k-1}|.$$

The proof of the analogous result based on the data $\{f(t_k), h(t_k)\}$ proceeds similarly.

□

For d_0 fixed and $d_1 > 0$, let $P_1(d_1)(u) = d_0\rho_0(u) + d_1\lambda_1(u)$ for $u \in J_1$. Then P_1 is a mapping from $[0, \infty]$ into a one-dimensional subspace of $W(J_1)$. It follows from (3.3) in the case $k = 1$ that

$$\langle \Delta D(u_2), {}^t\delta\Phi[P_1(d_1), P_1(d_0)](\theta)\rangle = \langle (d_1 - d_0)\lambda_1(u_2), {}^t\delta\Phi\,[P_1(d_1), P_1(d_0)](\theta)\rangle$$
$$= (d_1 - d_0) \int_0^{T_1}\int_0^{x_0(t)} \lambda_1(u_2)\partial_x u_2 \partial_x \phi \, \mathrm{d}x \, \mathrm{d}t.$$

This means that the double integral in the expression above is a representation for the derivative with respect to the parameter d, of the coefficient-to-data mapping, Φ, restricted to the one-dimensional subspace of $W(J_1)$. Since the double integral can be shown to be nonzero, it follows that the restricted input/output mapping is locally approximately invertible. Lemma 3.1 asserts that, if we are given the data, $\{f(t_k), g(t_k)\}$ or $\{f(t_k), h(t_k)\}$, then we can compute the nodal values $\{d_k\}$ which reproduce the measured data in the sense of (3.3) and that these nodal values approach the nodal values of the 'true coefficient' $D(u_1)$, as the mesh size of the outer mesh decreases. However, this conclusion ignores certain difficulties:

- it is not possible to know the coefficient $D_1(\mu(x, t))$ in the adjoint problems since D_1 is the coefficient we wish to identify and μ is an indeterminate point between u_1 and u_2. This means we can only approximate the solution to the adjoint problem and this will have an influence on the conclusions of lemma 3.1.
- the integrals in the identity can only be approximated by numerical integrations for which only a limited degree of refinement is possible. This may further interfere with the agreement between d_k and $D(\mu_k)$.

We will consider both of these effects, starting with the effect of the approximate adjoint solution.

Note first, that the algorithm (3.3) asserts that in determining the nodal value μ_k, it is necessary to solve the adjoint problem only on the strip $S_k = \{(0, 1) \times (T_{k-1}, T_k)\}$. Let $\hat{\phi}(x, t)$ denote the adjoint solution we compute using a convenient approximation for the unknown coefficient $D_1(\mu(x, t))$ on this strip. For example, suppose the coefficient in the g-adjoint problem is chosen to have the known constant value, d_{k-1}, i.e.,

$$D_1(\mu(x, t)) = d_{k-1}, \qquad \mu(x, t) \in J_k = [\mu_{k-1}, \mu_k].$$

Then if we replace ϕ in (3.3) by $\hat{\phi}(x, t)$, we can denote the resulting computed nodal value by $\hat{d}_k$. Note that with this choice for the coefficient, there is now no difficulty in solving the adjoint problem (2.6) for $\hat{\phi}$ on the strip, $(0, 1) \times [T_{k-1}, T_k]$ and proceeding to compute $\hat{d}_k$ using (3.3). It remains to be seen how the values $\hat{d}_k$ compare to the values d_k. We begin with a lemma.

Lemma 3.2. *Let $f(t)$ satisfy (2.2), let coefficient D satisfy (2.3) and let Π denote a fixed partition, $\Pi = \{\mu_k = f(T_k) : k = 0, 1, \ldots, M\}$ of J. For k between 1 and M consider the following adjoint problem,*

$$\begin{aligned}
&\partial_t \phi(x, t) + c\partial_{xx}\phi(x, t) = 0 && \in S_k, \\
&\phi(x, T_k) = 0 && x \in (0, 1), \\
&\phi(0, t) = f(T_k - t) && t \in (T_{k-1}, T_k), \\
&\partial_x \phi(1, t) = 0 && t \in (T_{k-1}, T_k).
\end{aligned}$$

Suppose $\{\phi_i, c_i\}, i = 1, 2$ denote two solutions to the adjoint problem corresponding to distinct choices of the coefficient c. In particular, suppose $\phi_1 = \phi(x, t, c_1, \theta)$ for the constant $c_1 = d_{k-1}$, while $\phi_2 = \phi(x, t, c_2, \theta)$ corresponding to the choice, $c_2(x, t) = D(\mu(x, t))$, where $\mu(x, t)$ denotes a function that is continuous on the strip $S_k = (0, 1) \times (T_{k-1}, T_k)$ with values in $J_k = [\mu_{k-1}, \mu_k]$. Then

$$\|\partial_x(\phi_1 - \phi_2)\|_{L^2(S_k)} \leqslant C|\mu_k - \mu_{k-1}|.$$

Proof. Begin by noting that $\Delta\phi = \phi_1 - \phi_2$ satisfies

$$\begin{aligned} &\partial_t(\Delta\phi) + c_1\partial_{xx}(\Delta\phi) = (c_2 - c_1)\partial_{xx}\phi_2 && (x,t) \in S_k, \\ &\Delta\phi(x, T_k) = 0 && x \in (0,1), \\ &\Delta\phi(0,t) = 0 && \in (T_{k-1}, T_k), \\ &\partial_x(\Delta\phi)(1,t) = 0 && t \in (T_{k-1}, T_k), \end{aligned}$$

and if ψ denotes an arbitrary test function, then

$$\int\int_{S_k} \{\partial_t(\Delta\phi) + c_1\partial_{xx}(\Delta\phi)\}\partial_x\psi \,\mathrm{d}x\,\mathrm{d}t = \int\int_{S_k} \{-\Delta c\partial_{xx}\phi_2\}\partial_x\psi \,\mathrm{d}x\,\mathrm{d}t.$$

Integration by parts yields

$$\int\int_{S_k} \partial_t(\Delta\phi)\partial_x\psi \,\mathrm{d}x\,\mathrm{d}t = \int\int_{S_k} \partial_x(\Delta\phi)\partial_t\psi \,\mathrm{d}x\,\mathrm{d}t + \int_0^1 \Delta\phi\partial_x\psi|_{t=0}^{t=T}\,\mathrm{d}x - \int_{T_{k-1}}^{T_k} \Delta\phi\partial_x\psi|_{x=0}^{x=1}\,\mathrm{d}t,$$

and

$$\int\int_{S_k} \partial_{xx}(\Delta\phi)\partial_x\psi \,\mathrm{d}x\,\mathrm{d}t = -\int\int_{S_k} \partial_x(\Delta\phi)\partial_{xx}\psi \,\mathrm{d}x\,\mathrm{d}t + \int_{T_{k-1}}^{T_k} \partial_x(\Delta\phi)\partial_x\psi|_{x=0}^{x=1}\,\mathrm{d}t,$$

so

$$\begin{aligned} &\int\int_{S_k} \partial_x(\Delta\phi)[\partial_t\psi - c_1\partial_{xx}\psi]\,\mathrm{d}x\,\mathrm{d}t + \int_0^1 \Delta\phi\partial_x\psi|_{t=T_{k-1}}^{t=T_k}\,\mathrm{d}x \\ &\qquad - \int_{T_{k-1}}^{T_k} \Delta\phi\partial_x\psi|_{x=0}^{x=1}\mathrm{d}t + c_1\int_{T_{k-1}}^{T_k} \partial_x(\Delta\phi)\partial_x\psi|_{x=0}^{x=1}\,\mathrm{d}t \\ &\quad = \int\int_{S_k} \{-\Delta c\partial_{xx}\phi_2\}\,\partial_x\psi\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Now choose the test function ψ to satisfy

$$\begin{aligned} &\partial_t\psi - c_1\partial_{xx}\psi = \partial_x(\Delta\phi) && (x,t) \in S_k, \\ &\psi(x, T_{k-1}) = 0 && x \in (0,1), \\ &\partial_x\psi(0,t) = 0 && t \in (T_{k-1}, T_k), \\ &\psi(1,t) = 0 && t \in (T_{k-1}, T_k). \end{aligned}$$

Then the previous integral identity reduces to

$$\int\int_{S_k} [\partial_x(\Delta\phi)]^2\,\mathrm{d}x\,\mathrm{d}t = \int\int_{S_k} (c_2 - c_1)\partial_{xx}\phi_2\partial_x\psi\,\mathrm{d}x\,\mathrm{d}t.$$

Now, ψ is the solution to a linear problem with constant coefficients so it can be expressed in terms of a Green's function, $\Gamma(x,t)$,

$$\psi(x,t) = \int_{T_{k-1}}^{t}\int_0^1 \Gamma(x-y, t-\tau)\partial_x(\Delta\phi)(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau,\ (x,t) \in S_k,$$

and

$$\partial_x\psi(x,t) = \int_{T_{k-1}}^{t}\int_0^1 \partial_x\Gamma(x-y, t-\tau)\partial_x(\Delta\phi)(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau.$$

Then for all $(x, t) \in S_k$,

$$|\partial_x \psi(x,t)| \leqslant \int_{T_{k-1}}^{t} \int_0^1 |\partial_x \Gamma(x-y, t-\tau)\partial_x(\Delta\phi)(y,\tau)|\, dy\, d\tau$$
$$\leqslant \left(\int_{T_{k-1}}^{T_k} \int_0^1 |\partial_x \Gamma(x-y, t-\tau)|^2\, dy\, d\tau \right)^{1/2}$$
$$\times \left(\int_{T_{k-1}}^{T_k} \int_0^1 |\partial_x (\Delta\phi)(y,\tau)|^2\, dy\, d\tau \right)^{1/2}$$

and

$$\max_{(x,t)\in S_k} |\partial_x \psi(x,t)| \leqslant C \|\partial_x(\Delta\phi)\|_{L^2(S_k)}.$$

Then it follows that

$$\int\int_{S_k} [\partial_x(\Delta\phi)]^2\, dx\, dt = \left| \int\int_{S_k} (c_2 - c_1)\partial_{xx}\phi_2 \partial_x \psi\, dx\, dt \right|$$
$$\leqslant \max_{S_k} |\Delta c(x,t)| \int\int_{S_k} |\partial_{xx}\phi_2 \partial_x \psi|\, dx\, dt$$
$$\leqslant \max_{S_k} |\Delta c(x,t)| \|\partial_{xx}\phi_2\|_{L^1} \|\partial_x \psi\|_\infty$$

and

$$\|\partial_x(\Delta\phi)\|_{L^2(S_k)} \leqslant C \max_{S_k} |\Delta c(x,t)|.$$

Also

$$\max_{S_k} |\Delta c(x,t)| = \max_{S_k} |d_{k-1} - D(\mu(x,t))|$$
$$\leqslant |d_{k-1} - D(\mu_{k-1})| + \max_{S_k} |D(\mu_{k-1}) - D(\mu(x,t))|$$
$$\leqslant 2K|\mu_k - \mu_{k-1}|.$$

Then, it follows that,

$$\|\partial_x(\Delta\phi)\|_{L^2(S_k)} \leqslant C|\mu_k - \mu_{k-1}|. \qquad \square$$

Now we will use this estimate in considering the effect of using the approximate adjoint solution in the determination of the first nodal value, d_1. It follows from (3.3) that the difference between the value, d_1, computed using the correct but unknown adjoint solution and the value, $\hat{d}_1$, computed using an incorrect but computable adjoint solution is given by,

$$\hat{d}_1 - d_1 = \frac{\int_0^{T_1} (g(t) - g_2(t))\theta(t)\, dt}{\int_0^{T_1} \int_0^{x_0(t)} \lambda_1(u_2)\partial_x u_2 \partial_x \hat{\phi}\, dx\, dt} - \frac{\int_0^{T_1} (g(t) - g_2(t))\theta(t)\, dt}{\int_0^{T_1} \int_0^{x_0(t)} \lambda_1(u_2)\partial_x u_2 \partial_x \phi\, dx\, dt}$$
$$= \frac{(g - g_2, \theta)}{II(\hat{\phi})} - \frac{(g - g_2, \theta)}{II(\phi)}$$
$$= (g - g_2, \theta) \left\{ \frac{1}{II(\hat{\phi})} - \frac{1}{II(\phi)} \right\}$$
$$\hat{d}_1 - d_1 = (d_1 - d_0) \left\{ \frac{II(\phi) - II(\hat{\phi})}{II(\hat{\phi})} \right\}.$$

Here

$$II(\hat{\phi}) = \int_0^{T_1} \int_0^{x_0(t)} \lambda_1(u_2)\partial_x u_2 \partial_x \hat{\phi}\, dx\, dt.$$

We wish to show that as the outer mesh is refined, the discrepancy $II(\phi) - II(\hat{\phi})$ that is due to solving the adjoint problem with the wrong coefficient decreases to zero. On the other hand, $II(\hat{\phi})$ also decreases towards zero as the mesh is refined. To see whether $II(\hat{\phi})$ decreases more or less rapidly than $II(\phi) - II(\hat{\phi})$, it is necessary to examine the asymptotic behaviour of $II(\hat{\phi})$. We assume that $x_0(T_1) < 1$ since if this is not the case, we can always refine the outer partition to shrink the width of the strip S_1 so as to make it true. Then the domain of integration for $II(\hat{\phi})$ is the approximately triangular region $\{0 \leqslant x \leqslant x_0(t), 0 \leqslant t \leqslant T_1\}$. An exact analysis of the asymptotic rate of convergence of $II(\hat{\phi})$ as T_1 tends to zero is difficult, but if we assume that $f(t) = At$ for a positive constant A, then it is possible to solve explicitly for $u_2(x,t)$ and $\hat{\phi}(x,t)$. Using arguments such as in [1], one finds that $g(t) = -D(u_1(0,t))\partial_x u_1(0,t)$ and $g_2(t) = -d_k \partial_x u_2(0,t)$ behave asymptotically like $\sqrt{t}$.

This leads to

$$\int_0^{T_1} (g(t) - g_2(t))\theta(t)\,\mathrm{d}t = \int_0^{T_1} (g(t) - g_2(t))A(T_1 - t)\,\mathrm{d}t \approx C T_1^{5/2}. \tag{3.5}$$

A similar crude estimate for $\partial_x u_2 \partial_x \hat{\phi}$ on $0 \leqslant x \leqslant 1, 0 \leqslant t \leqslant T_1$, is the following:

$$\partial_x u_2 \partial_x \hat{\phi}(x,t) \approx \sqrt{t}m(x)\sqrt{T_1 - t}m(x),$$

where $m(x)$ denotes a decreasing function with $m(0) = 1$ and $m(1) = 0$. In addition, for T_1 small, one can suppose $x_0(t) \approx at$ for a positive constant a, and this leads to

$$\begin{aligned} II(\hat{\phi}) &= \int_0^{T_1} \int_0^{x_0(t)} \lambda_1(u_2)\partial_x u_2 \partial_x \hat{\phi}\,\mathrm{d}x\,\mathrm{d}t \\ &\approx \int_0^{T_1} \int_0^{at} \frac{u_2(x,t)}{AT_1} \sqrt{t}m(x)\sqrt{T_1 - t}m(x)\,\mathrm{d}x\,\mathrm{d}t \end{aligned}$$

i.e.

$$II(\hat{\phi}) \approx C T_1^{5/2}. \tag{3.6}$$

Since this estimate (3.6) is rather rough, the quantity $II(\hat{\phi})$ was computed numerically for a sequence of values for T_1 decreasing to zero. The result of this numerical asymptotic estimate supported the estimate (3.6) which asserts that $II(\hat{\phi})$ decreases like the $\frac{5}{2}$ power of T_1 as T_1 tends to zero.

Now

$$\hat{d}_1 - d_1 = (d_1 - d_0)\left\{\frac{II(\phi) - II(\hat{\phi})}{II(\hat{\phi})}\right\},$$

and

$$\begin{aligned} |II(\phi) - II(\hat{\phi})| &= \left|\int_0^{T_1}\int_0^{x_0(t)} \lambda_1(u_2)\partial_x u_2[\partial_x\phi - \partial_x\hat{\phi}]\,\mathrm{d}x\,\mathrm{d}t\right| \\ &\leqslant C(T_1)\|\partial_x(\Delta\phi)\|_{L^2(S_1)} \leqslant C(T_1)|\mu_1 - \mu_0|. \end{aligned}$$

Also,

$$|d_1 - d_0| = |D(\mu_1) - D(\mu_0)| \leqslant K|\mu_1 - \mu_0|,$$

and hence

$$|\hat{d}_1 - d_1| \leqslant |d_1 - d_0|\left|\frac{II(\phi) - II(\hat{\phi})}{II(\hat{\phi})}\right| \leqslant \frac{KC(T_1)}{II(\hat{\phi})}|\mu_1 - \mu_0|^2.$$

Then for T_1 sufficiently small,

$$|\hat{d}_1 - d_1| \leqslant \frac{KC(T_1)}{CT_1^{5/2}}|\mu_1 - \mu_0|^2$$
$$\leqslant \frac{Kf'(\tau)^2}{C}T_1^{-1/2} \qquad \text{for some} \quad \tau > 0.$$

In general, we have

Lemma 3.3. *For $f(t) = At, A > 0$, for coefficient D satisfying (2.3) and for a fixed partition, $\Pi = \{\mu_k = AT_k : k = 0, 1, \ldots, M\}$ of J, fix k between 1 and M. Let $\hat{\phi} = \phi(x, t, d_{k-1}, A(T_k - t))$ and $\phi = \phi(x, t, c, A(T_k - t))$ corresponding to the coefficients, d_{k-1} and $c(x, t) = D(\mu(x, t))$, respectively. Finally, let $\hat{d}_k$ and d_k denote the nodal values determined from (3.3) using the values $[d_0, d_1, \ldots, d_{k-1}]$ and the adjoint solutions $\hat{\phi}$ and ϕ, respectively. Then*

$$|\hat{d}_k - d_k| \leqslant \frac{K}{II(\hat{\phi})}|\mu_k - \mu_{k-1}|^2 \leqslant \frac{Kf'(\tau)^2}{C}|T_k - T_{k-1}|^{-1/2}.$$

This lemma implies that the error introduced into the identification by solving the adjoint problem with an approximate coefficient has an increasing effect as the outer mesh is refined. As the mesh is refined, the discrepancy $II(\phi) - II(\hat{\phi})$ does tend to zero like the square of the mesh size. However, as the mesh size tends to zero, we find also that $II(\hat{\phi})$, which can be viewed as an approximation to the Gateaux derivative of the mapping Φ restricted to a one-dimensional subspace of $W(J_k)$, tends to zero even faster, (like the $\frac{5}{2}$ power of the mesh size). It is likely that the means of approximating the adjoint solution could be improved so that $II(\phi) - II(\hat{\phi})$ would approach zero sufficiently rapidly that $|\hat{d}_k - d_k|$ would tend to zero as the mesh size goes to zero. However, the next result will show that such an improvement does not improve the convergence of the approximate solution.

We wish finally to consider the effect of numerical integration errors on the calculation of $\hat{d}_k$. We begin by considering $k = 1$. We have

$$\hat{d}_1 = d_0 + \frac{\int_0^{T_1}(g(t) - g_2(t))\theta(t)\,\mathrm{d}t}{\int_0^{T_1}\int_0^{x_0(t)}\lambda_1(u_2)\partial_x u_2 \partial_x \hat{\phi}\,\mathrm{d}x\,\mathrm{d}t} = d_0 + \frac{I(g - g_2)}{II(\hat{\phi})},$$

and

$$\hat{d}_1^* = d_0 + \frac{I^*(g - g_2)}{II^*(\hat{\phi})},$$

where $I^*(g - g_2)$ and $II^*(\hat{\phi})$ denote, respectively, the computed results using the inner mesh to numerically approximate the corresponding exact single and double integrals. Then,

$$\hat{d}_1^* = d_0 + \frac{I^*(g - g_2) - I(g - g_2) + I(g - g_2)}{II^*(\hat{\phi}) - II(\hat{\phi}) + II(\hat{\phi})}$$
$$= d_0 + \frac{I(g - g_2)}{II(\hat{\phi})}\frac{1 + \varepsilon_1}{1 + \varepsilon_2},$$

where

$$\varepsilon_1 = \left|\frac{I - I^*}{I}\right| \qquad \text{and} \qquad \varepsilon_2 = \left|\frac{II - II^*}{II}\right|.$$

Now

$$\frac{1 + \varepsilon_1}{1 + \varepsilon_2} \approx 1 + \varepsilon_1 + \varepsilon_2,$$

so

$$\hat{d}_1^* = d_0 + \frac{I(g-g_2)}{II(\hat{\phi})}\frac{1+\varepsilon_1}{1+\varepsilon_2} \approx d_0 + \frac{I(g-g_2)}{II(\hat{\phi})}(1+\varepsilon_1+\varepsilon_2),$$

and

$$|\hat{d}_1 - \hat{d}_1^*| \leqslant \left|\frac{I(g-g_2)}{II(\hat{\phi})}\right|(\varepsilon_1+\varepsilon_2) = |\hat{d}_1 - d_0|(\varepsilon_1+\varepsilon_2).$$

The numerical integration errors are estimated by terms of the form

$$|I - I^*| \leqslant C(\Delta t)^2 \qquad \text{for} \quad \Delta t = \text{inner mesh size},$$

and

$$|II - II^*| \leqslant C(\Delta x \Delta t) = C(\Delta t)^2.$$

Use of higher order integration schemes is limited by the fact that reducing the mesh size of the outer or J-mesh in order to achieve identification accuracy absorbs I-mesh node points into the J-mesh leaving only enough points in the inner mesh to perform low-order numerical integrations.

It follows from (3.5) and (3.6) that

$$I = \int_0^{T_1} (g(t) - g_2(t))A(T_1 - t)\,\mathrm{d}t \approx T_1^{5/2},$$
$$II = \int_0^{T_1}\int_0^{x_0(t)} \lambda_1(u_2)\partial_x u_2 \partial_x \phi\,\mathrm{d}x\,\mathrm{d}t \approx T_1^{5/2}.$$

Then, since $T_1 = k\Delta t$, we find

$$|\hat{d}_1 - \hat{d}_1^*| \leqslant \left|\frac{I(g-p_M)}{II(\hat{\phi})}\right|(\varepsilon_1+\varepsilon_2)$$
$$\leqslant |\hat{d}_1 - d_0|\frac{C_1(\Delta t)^2}{C_2(k\Delta t)^{5/2}} \leqslant C(\Delta t)^{-1/2}.$$

More generally, we have

Lemma 3.4. *Under the conditions of (3.3), let $\hat{d}_k^*$ reflect the error induced in $\hat{d}_k$ by numerically approximating the integrals needed for (3.3). Then, as the (inner and outer) mesh size tends to zero,*

$$|\hat{d}_k - \hat{d}_k^*| \leqslant C(\Delta t)^{-1/2}.$$

This estimate suggests that as the outer mesh is refined in order to improve the accuracy of the identification of the nodal values of $D(u_1)$, more and more node points of the inner mesh are absorbed into the outer mesh, resulting in numerical integration errors, $|I - I^*|$ and $|II - II^*|$, that are of order Δt^2. At the same time, the approximate Gateaux derivative $II(\hat{\phi})$ tends to zero like $\Delta t^{5/2}$ so the effect of approximating the integrals becomes magnified as Δt tends to zero. Evidently, at some point the values of the integrals used to compute d_k become of the same order of magnitude as the numerical integration errors and the computation then no longer contains information. Further decreasing the mesh size then only increases the error.

Finally, we can combine lemmas 3.1, 3.3 and 3.4 to write

$$|D(\mu_k) - \hat{d}_k^*| = |D(\mu_k) - d_k + d_k - \hat{d}_k + \hat{d}_k - \hat{d}_k^*|$$
$$\leqslant |D(\mu_k) - d_k| + |d_k - \hat{d}_k| + |\hat{d}_k - \hat{d}_k^*|,$$

and

$$|D(\mu_k) - \hat{d}_k^*| \leqslant C_1 \Delta t + C_2 (\Delta t)^{-1/2}. \tag{3.7}$$

Evidently the error in identifying d_k does not tend to zero as Δt tends to zero but is minimized by an optimal Δt different from zero.

4. Numerical experiments

In the numerical experiments we describe here, we chose $f(t) = At$ for some positive constant A and defined the node points μ_k for the outer mesh by $\mu_k = AT_k, k = 0, 1, \ldots, M$. Here, for each k, $T_k = t_j$ for some $j > k$ where $0 = t_0 < t_1 < \cdots < t_N = T$ denotes the (inner) partition of $[0, T]$. The unknown nodal values for the coefficient $D(u)$ are given by $d_k = D(\mu_k)$, and we assume that d_0 is known. Since the initial state for the direct problem, $u(x, 0)$, is constant and $f(t)$ is monotone increasing, the domain Q_T consists of a sequence of non-overlapping strips, S_k, with only one nodal value active on each strip. The algorithm to identify D from the (f, g)-data then proceeds as follows.

The algorithm. To begin, we apply the g-integral identity (2.7) on Q_1. Since the solution of the direct problem satisfies lemma 2.1(a), we have $\mu_0 \leqslant u_1(x, t) \leqslant \mu_1$ for $(x, t) \in Q_1$. Then only the known nodal value d_0 and the unknown nodal value d_1 are active on this strip. We are going to compute the unknown nodal values iteratively and denote the ith iteration for d_k by $d_k^{(i)}$. We set $d_1^{(0)} = d_0$.

We apply the integral identity (2.7) on Q_1 with

$$D_1 = P_1\big[d_0, d_1^{(1)}\big] \qquad \text{and} \qquad D_2 = P_1\big[d_0, d_1^{(0)}\big],$$
$$u_2(x, t) = u(x, t; D_2, At) \qquad \text{and} \qquad g_2(t) = \Phi[f, D_2],$$
$$\hat{\phi}(x, t) = \phi(x, t; D_2, A(T_1 - t)).$$

We compute

$$A_{11} = \int_0^{T_1} \int_0^{x_0(t)} \lambda_1(u_2) \partial_x u_2 \partial_x \hat{\phi} \, \mathrm{d}x \, \mathrm{d}t, \qquad b_1 = \int_0^{T_1} (g(t) - g_2(t)) A(T_1 - t) \, \mathrm{d}t,$$

and solve

$$A_{11}\big(d_1^{(1)} - d_0\big) = b_1.$$

Note that A_{11} and b_1 are computed from $u_2, \hat{\phi}, g_2$ all of which are based on the known coefficient D_2.

To continue, we apply the g-integral identity (2.7) first on Q_1, where only d_0, d_1 are active, and then apply the g-integral identity (2.7) again, but now on Q_2 where d_0, d_1, d_2 are active. That is,
on Q_1

$$D_1 = P_1\big[d_0, d_1^{(2)}\big] \qquad d_1^{(2)} \text{ is unknown,}$$

and

$$D_2 = P_1\big[d_0, d_1^{(1)}\big] \qquad d_1^{(1)} \text{ is known,}$$

and we compute A_{11} and b_1 as before.

Note that $u_2, \hat{\phi}, g_2$ are based on the updated coefficient D_2 so that, in general, $d_1^{(2)}$ will not be the same as $d_1^{(1)}$.

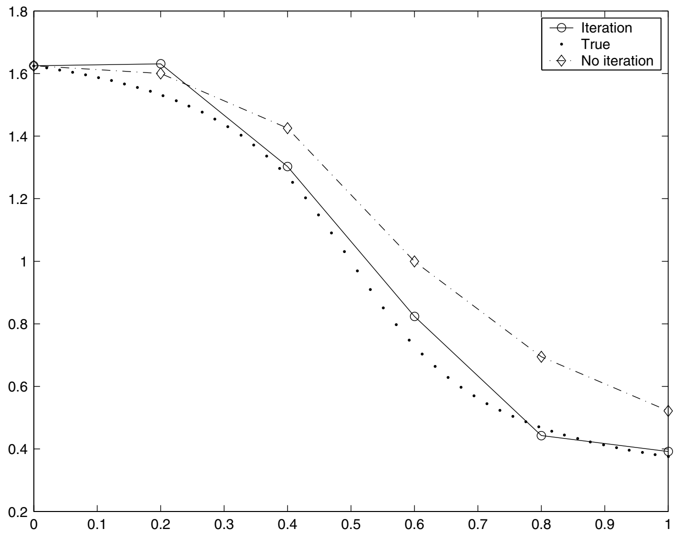

Figure 2. Recovery of $D(u) = 2 - \arctan\,[6(u - \frac{1}{2})]$.

On Q_2

$$D_1 = P_2\big[d_0, d_1^{(2)}, d_2^{(1)}\big]$$

and

$$D_2 = P_2\big[d_0, d_1^{(1)}, d_2^{(0)}\big] \qquad \text{note : } d_2^{(0)} = d_1^{(1)},$$

we compute

$$A_{2,1} = \int\int_{Q_{21}} \lambda_1(u_2)\partial_x u_2 \partial_x \hat{\phi}\,\mathrm{d}x\,\mathrm{d}t,$$

$$Q_{21} = \{\mu_0 \leqslant u_2(x,t) \leqslant \mu_1, 0 \leqslant t \leqslant T_2\},$$

$$A_{2,1} = \int_{T_1}^{T_2}\int_0^{x_1(t)} \lambda_2(u_2)\partial_x u_2 \partial_x \hat{\phi}\,\mathrm{d}x\,\mathrm{d}t,$$

$$b_2 = \int_0^{T_2} (g(t) - g_2(t))A(T_2 - t)\,\mathrm{d}t,$$

and we solve

$$\begin{bmatrix} A_{11} & 0 \\ A_{2,1} & A_{2,2} \end{bmatrix} \begin{bmatrix} d_1^{(2)} - d_1^{(1)} \\ d_2^{(1)} - d_2^{(0)} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

We proceed in this way, where at the kth stage we apply the integral identity k times, once on each of the strips Q_1 to Q_k. Of course this produces k equations, one for each strip. On each strip, Q_j, there are only j unknown active node values $d_1^{(p)}, \ldots, d_j^{(q)}$, at various stages of iteration, hence the jth equation contains only the first j unknowns. This leads to a k by k lower triangular system for the differences $d_j^{(p)} - d_j^{(p-1)}$. At the kth stage of the algorithm we are solving for the first iterate for d_k, for the second iterate of d_{k-1}, etc. This algorithm, which we will call the iterative algorithm, differs from the non-iterative algorithm described in the preceding section. The non-iterative algorithm amounts to suppressing the iterative feature so that for each k, the nodal value d_k is obtained by solving just a single equation,

$A_{kk}(d_k - d_{k-1}) = b_k$. Suppressing the iteration leads to cascading errors in the sequentially computed nodal values d_k as shown in figure 2. The coefficient shown in this figure

$$D(u) = 2 - \arctan[6(u - 1/2)] \qquad 0 < u < 1, \tag{4.1}$$

was recovered in two ways. In the first, the non-iterative algorithm was applied to the data $\{f, g\}$ to produce the dashed line plot, while the iterative algorithm was applied to produce the solid line plot. The data were generated by solving the direct problem (2.1) using a functional form of the coefficient (4.1) on a mesh of 70 nodes with the Matlab solver `ode15s`. The flux, $g(t)$, was then computed using a difference formula. These flux data were submitted to the recovery algorithms, which both used a 40-node mesh and `ode15s` to compute solutions to the direct and adjoint problem. It is clear from the figure that the errors in non-iterated nodal values for $D(u)$ accumulate as the values are sequentially determined. We point out that determining d_k we are obliged to integrate over the approximately triangular region $\{0 < x < x_k(t), T_{k-1} < t < T_k\}$. However, the algorithm must numerically approximate $x_0(t_j)$ on the inner mesh, and this leads to a systematic overestimation of the value of A_{kk} which, in turn, leads to a correction term that is too small. The fact that D is a decreasing function of u, as given in equation (4.1), leads to a negative $\Delta g(t)$ and a negative correction, b_k/A_{kk}. This is evident in the dashed-line plot of figure 2. The fact that A_{kk} is too large causes the negative correction to be too small so that the graph of the computed polygonal function lies above the graph of the true coefficient. Since the integrals for A_{kk} and b_k involve only the interval $[T_{k-1}, T_k]$, each identified value, d_k, can do nothing to diminish errors in previously identified values, hence the identification error accumulates.

This suggests that iteration might prove useful. The solid line plot in figure 2 shows the result of identifying the coefficient 4.1) but now iterating as follows. We use the identity (2.7) on Q_1 together with the known value, d_0, to identify $d_1^{(1)}$. Here the known value, d_0, is used to compute $u_2(x, t)$, $g_2(t)$ and $\hat{\phi}(x, t)$. Next we use the identity (2.7) on Q_1 and Q_2 together with known values, $d_0, d_1^{(1)}$ to identify $d_1^{(2)}$ and $d_2^{(1)}$. In the next step, we use the identity (2.7) on Q_1, Q_2 and Q_3 together with known values, d_0, $d_1^{(2)}$ and $d_2^{(1)}$ to identify $d_1^{(3)}$, $d_2^{(2)}$ and $d_3^{(1)}$. At each stage, the known nodal values are used to compute $u_2(x, t)$, $g_2(t)$ and $\hat{\phi}(x, t)$. Continuing in this way, we eventually obtain $d_M^{(1)}, d_{M-1}^{(2)}, \ldots, d_1^{(M)}$. It is evident from the solid line plot in figure 2 that as a result of the iteration, the errors no longer exhibit the cumulative character seen in the dashed line plot, where iteration was not applied.

Here the coefficient

$$D(u) = 1 + \tfrac{1}{2}\sin(2\pi u) \qquad 0 < u < 1, \tag{4.2}$$

was used to generate flux data as in the previous example, although here the Matlab solver `ode23s` was used. These data were passed to the iterative recovery algorithm, the results of which are plotted in figure 3.

The qualitative agreement between the computed and true coefficient appears reasonable in this figure. Note that the approximation initially lies above the plot of the true coefficient (4.2) in regions where D is increasing, which is in agreement with the analysis of the previous experiment. The value at the last node is not iterated in this scheme, and is visibly less accurate than the computed values on other nodes.

Figure 4 displays the effect of refining the outer mesh by increasing M, the number of nodes, in order to identify the coefficient

$$D(u) = 1 + u \qquad 0 < u < 1.$$

The results for $M = 2$, 5 and 9 are shown in addition to a plot of the L^2-error versus M. This last display shows the error decreasing with increasing M up to about $M = 5$, at which point the error again begins to increase. This result is in qualitative agreement with (3.7).

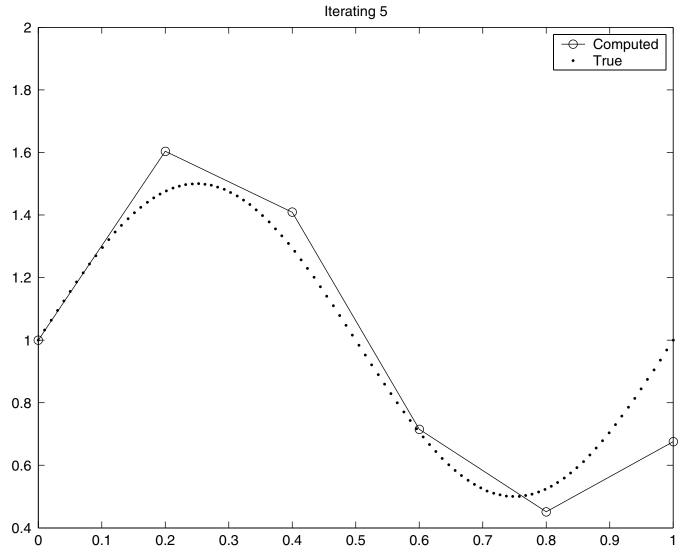

Figure 3. Iterative recovery of $D(u) = 1 + \frac{1}{2}\sin(2\pi u)$.

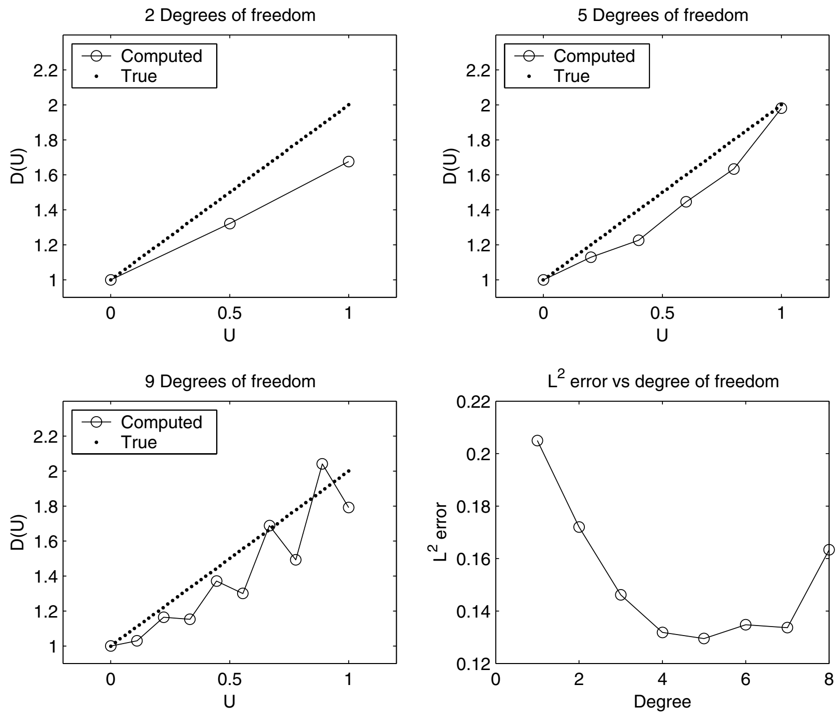

Figure 4. Recovery using 2, 5 and 9 degrees of freedom.

Figure 5 represents coefficient recovery in which the data contained induced error. A relative uniform random error of 10% was induced in the flux data, and the iterative algorithm was applied. The flux data used for recovery is plotted in figure 6. The recovered coefficient,

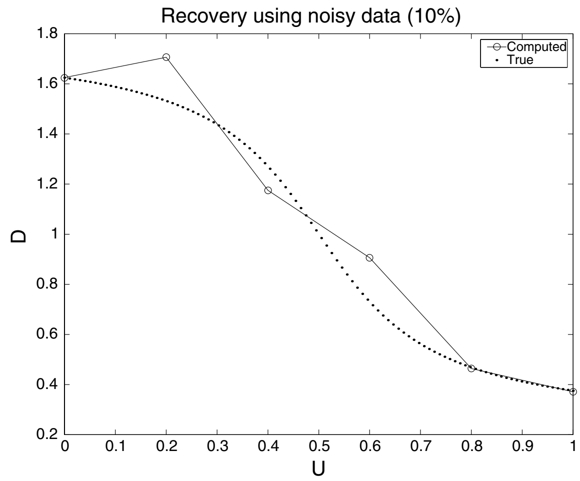

Figure 5. Recovery with noisy flux data.

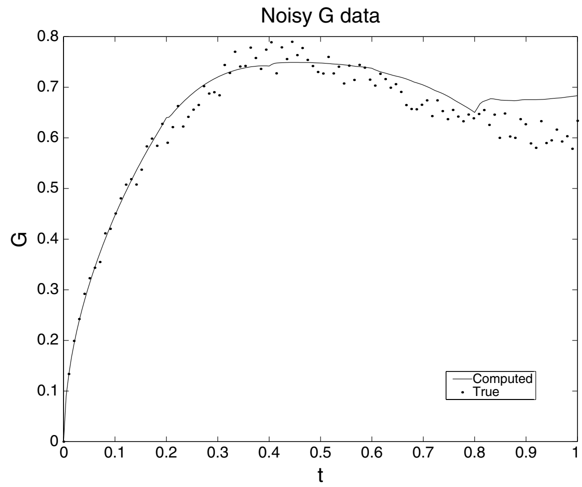

Figure 6. Noisy flux (G) data.

plotted in figure 5, appears to capture the general structure of the true coefficient. No preprocessing was applied to these data, which was possible since the error had mean zero. The integration of the g data in (3.3) allows much of this error to cancel.

5. Conclusions

The integral identities (2.7) and (2.9) are equations providing explicit representations for the input-to-output mappings associated with the inverse problem of identifying an unknown diffusion coefficient from overspecified data measured on the boundary. These equations provide a means for proving that the input output maps are continuous, injective and strictly monotone. Such information is not so readily obtained from an output least-squares approach nor from equation error techniques. The equations may also be the basis for a numerical approximation procedure although here it might be more difficult to compete with sophisticated OLS implementations.

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