# A Note on Extension of TOPSIS to Multiple Criteria Decision Making with Pythagorean Fuzzy Sets

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In this note, we point out an error to the proof of Theorem 3.4 in Zhang and Xu (Int J Intell Syst 2014;29(12):1061–1078) by a counterexample. We find that the inequality (i.e., $|(\pi_{\beta_1})^2 - (\pi_{\beta_2})^2| \leq |(\pi_{\beta_1})^2 - (\pi_{\beta_3})^2|$) with respect to the degrees of indeterminacy of any three Pythagorean fuzzy numbers in the proof of Theorem 3.4 in Zhang and Xu's paper is not valid. A new proof is provided in this note. © 2015 Wiley Periodicals, Inc.

## 1. INTRODUCTION

Yager[1,2] introduced a generalization of intuitionistic fuzzy set, which is referred to as Pythagorean fuzzy set (PFS), to handle the situation in which a membership degree and a nonmembership degree have a sum that is greater than one and therefore they are allowable as Pythagorean membership grades rather than allowable for the space of intuitionistic membership grades. Recently, Zhang and Xu[3] provided the general definition of PFSs. They defined some novel operational laws of PFSs and investigated their desirable properties. To facilitate the discussion of PFSs, the ordered pair composed of its membership and nonmembership degrees is usually denoted as a Pythagorean fuzzy number (PFN). Motived by the study of Szmidt and Kacprzyk,[4] Zhang and Xu further provided the concept of distance measure for PFNs and proved some of its important properties. In Section 3 of Zhang and Xu's paper, they proposed the following theorem.

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THEOREM 3.4. [3] *Let $\beta_j = P(\mu_{\beta_j}, \nu_{\beta_j})(j = 1, 2, 3)$ be three PFNs, if $\beta_1 \leq \beta_2 \leq \beta_3$, then $d(\beta_1, \beta_2) \leq d(\beta_1, \beta_3)$ and $d(\beta_2, \beta_3) \leq d(\beta_1, \beta_3)$.*

*Proof.* If $\beta_1 \leq \beta_2 \leq \beta_3$, according to Definition 2.3,[3] we obtain $\mu_{\beta_1} \leq \mu_{\beta_2} \leq \mu_{\beta_3}$ and $\nu_{\beta_1} \geq \nu_{\beta_2} \geq \nu_{\beta_3}$. Then, $|(\mu_{\beta_1})^2 - (\mu_{\beta_2})^2| \leq |(\mu_{\beta_1})^2 - (\mu_{\beta_3})^2|$, $|(\nu_{\beta_1})^2 - (\nu_{\beta_2})^2| \leq |(\nu_{\beta_1})^2 - (\nu_{\beta_3})^2|$ and

$$\begin{aligned}\left|\left(\pi_{\beta_1}\right)^2 - \left(\pi_{\beta_2}\right)^2\right| &= \left|1 - \left(\mu_{\beta_1}\right)^2 - \left(\nu_{\beta_1}\right)^2 - \left(1 - \left(\mu_{\beta_2}\right)^2 - \left(\nu_{\beta_2}\right)^2\right)\right| \\ &= \left|\left(\mu_{\beta_2}\right)^2 - \left(\mu_{\beta_1}\right)^2 + \left(\left(\nu_{\beta_2}\right)^2 - \left(\nu_{\beta_1}\right)^2\right)\right| \\ &\leq \left|1 - \left(\mu_{\beta_1}\right)^2 - \left(\nu_{\beta_1}\right)^2 - \left(1 - \left(\mu_{\beta_3}\right)^2 - \left(\nu_{\beta_3}\right)^2\right)\right| \\ &= \left|\left(\mu_{\beta_3}\right)^2 - \left(\mu_{\beta_1}\right)^2 + \left(\left(\nu_{\beta_3}\right)^2 - \left(\nu_{\beta_1}\right)^2\right)\right| \\ &= \left|\left(\pi_{\beta_1}\right)^2 - \left(\pi_{\beta_3}\right)^2\right|\end{aligned}$$

Thus

$$\begin{aligned}d\left(\beta_1, \beta_2\right) &= \frac{1}{2}\left(\left|\left(\mu_{\beta_1}\right)^2 - \left(\mu_{\beta_2}\right)^2\right| + \left|\left(\nu_{\beta_1}\right)^2 - \left(\nu_{\beta_2}\right)^2\right| + \left|\left(\pi_{\beta_1}\right)^2 - \left(\pi_{\beta_2}\right)^2\right|\right) \\ &\leq \frac{1}{2}\left(\left|\left(\mu_{\beta_1}\right)^2 - \left(\mu_{\beta_3}\right)^2\right| + \left|\left(\nu_{\beta_1}\right)^2 - \left(\nu_{\beta_3}\right)^2\right| + \left|\left(\pi_{\beta_1}\right)^2 - \left(\pi_{\beta_3}\right)^2\right|\right) \\ &= d\left(\beta_1, \beta_3\right)\end{aligned}$$

Analogously, we can also prove $d(\beta_2, \beta_3) \leq d(\beta_1, \beta_3)$, which completes the proof of this theorem. □

In this paper, we study the proof of Theorem 3.4 in the literature[3] and present a revised version for it. This paper is organized as follows. In Section 2, we present some notations and definitions. In Section 3, we point out the problem of Zhang and Xu's proof and then we provide the correct version for it. In Section 4, we close this paper with some concluding remarks.

## 2. PRELIMINARIES

We begin with some basic concepts that are needed for the rest of this paper. For more details, we refer to the literiture.[3]

DEFINITION 2.1.[3] *Let a setX be a universe of discourse. A PFS P is an object having the form*

$$P = \{\langle x, P\left(\mu_P(x), v_P(x)\right)\rangle \,|\, x \in X\},$$

*where the function $\mu_P : X \to [0, 1]$ defines the degree of membership and $v_P : X \to [0, 1]$ defines the degree of nonmembership of element $x \in X$ to P,* *respectively, and for every* $x \in X$, *it holds that*

$$(\mu_P(x))^2 + (v_P(x))^2 \leq 1.$$

DEFINITION 2.2.[3] *For any PFS* $P$ *and* $x \in X$, $\pi_P(x) = \sqrt{1 - \mu_P(x)^2 - v_P(x)^2}$ *is called the degree of indeterminacy of* $x \in X$ *to* $P$. *For simplicity, we call* $P(\mu_P(x), v_P(x))$ *a PFN denoted by* $\beta = \langle \mu_\beta, \nu_\beta \rangle$, *where* $\mu_\beta, \nu_\beta \in [0, 1]$, $\pi_\beta = \sqrt{1 - (\mu_\beta)^2 - (\nu_\beta)^2}$, *and* $(\mu_\beta)^2 - (\nu_\beta)^2 \leq 1$.

DEFINITION 2.3.[3] *Let* $\beta_i = \langle \mu_{\beta_i}, \nu_{\beta_i} \rangle (i = 1, 2)$ *be two PFNs, a nature quasiordering on the PFNs is defined as follows:* $\beta_1 \geq \beta_2$ *if and only if* $\mu_{\beta_1} \geq \mu_{\beta_2}$ *and* $v_{\beta_1} \leq v_{\beta_2}$.

DEFINITION 2.4.[3] *Let* $\beta_i = \langle \mu_{\beta_i}, \nu_{\beta_i} \rangle (i = 1, 2)$ *be two PFNs, then we define the distance between* $\beta_1$ *and* $\beta_2$ *as follows:*

$$d(\beta_1, \beta_2) = \frac{1}{2}\left(\left|\left(\mu_{\beta_1}\right)^2 - \left(\mu_{\beta_2}\right)^2\right| + \left|\left(\nu_{\beta_1}\right)^2 - \left(\nu_{\beta_2}\right)^2\right| + \left|\left(\pi_{\beta_1}\right)^2 - \left(\pi_{\beta_2}\right)^2\right|\right).$$

## 3. ERROR IN ZHANG AND XU'S PROOF OF THEOREM 3.4 AND A NEW PROOF

Theorem 3.4 in the literature[3] describes that the following inequality is valid:

$$\left|\left(\pi_{\beta_1}\right)^2 - \left(\pi_{\beta_2}\right)^2\right| \leq \left|\left(\pi_{\beta_1}\right)^2 - \left(\pi_{\beta_3}\right)^2\right|. \tag{3.1}$$

However, the following counterexample illustrates that the inequality (3.1) does not hold. Let us consider three PFNs $\beta_1 = \langle 0.1, 0.9 \rangle$, $\beta_2 = \langle 0.2, 0.3 \rangle$, $\beta_3 = \langle 0.5, 0.2 \rangle$, then we obtain $\beta_1 \leq \beta_2 \leq \beta_3$ according to Definition 2.3. Following Definitions 2.1 and 2.2, we have

$$\begin{aligned}\left|\left(\pi_{\beta_1}\right)^2 - \left(\pi_{\beta_2}\right)^2\right| &= \left|\left(1 - \left(\mu_{\beta_1}\right)^2 - \left(v_{\beta_1}\right)^2\right) - \left(1 - \left(\mu_{\beta_2}\right)^2 - \left(v_{\beta_2}\right)^2\right)\right| \\ &= |0.18 - 0.87| = 0.69\end{aligned}$$

and

$$\begin{aligned}\left|\left(\pi_{\beta_1}\right)^2 - \left(\pi_{\beta_3}\right)^2\right| &= \left|\left(1 - \left(\mu_{\beta_1}\right)^2 - \left(v_{\beta_1}\right)^2\right) - \left(1 - \left(\mu_{\beta_3}\right)^2 - \left(v_{\beta_3}\right)^2\right)\right| \\ &= |0.18 - 0.71| = 0.53.\end{aligned}$$

It is obvious that $|(\pi_{\beta_1})^2 - (\pi_{\beta_2})^2| \geq |(\pi_{\beta_1})^2 - (\pi_{\beta_3})^2|$. This result violates the proof of Theorem 3.4 in the literature[3].

In what follows, we prove Theorem 3.4 in the literature[3] all over again.

*Proof.* According to Definition 2.4, we have

$$d(\beta_1, \beta_2) = \frac{1}{2}\left(\left|(\mu_{\beta_1})^2 - (\mu_{\beta_2})^2\right| + \left|(\nu_{\beta_1})^2 - (\nu_{\beta_2})^2\right| + \left|(\pi_{\beta_1})^2 - (\pi_{\beta_2})^2\right|\right)$$
$$= \frac{1}{2}\left(\left|(\mu_{\beta_1})^2 - (\mu_{\beta_2})^2\right| + \left|(\nu_{\beta_1})^2 - (\nu_{\beta_2})^2\right| + \left|(\mu_{\beta_2})^2 + (\nu_{\beta_2})^2 - \left((\mu_{\beta_1})^2 + (\nu_{\beta_1})^2\right)\right|\right),$$

and

$$d(\beta_1, \beta_3) = \frac{1}{2}\left(\left|(\mu_{\beta_1})^2 - (\mu_{\beta_3})^2\right| + \left|(\nu_{\beta_1})^2 - (\nu_{\beta_3})^2\right| + \left|(\pi_{\beta_1})^2 - (\pi_{\beta_3})^2\right|\right)$$
$$= \frac{1}{2}\left(\left|(\mu_{\beta_1})^2 - (\mu_{\beta_3})^2\right| + \left|(\nu_{\beta_1})^2 - (\nu_{\beta_3})^2\right| + \left|(\mu_{\beta_3})^2 + (\nu_{\beta_3})^2 - \left((\mu_{\beta_1})^2 + (\nu_{\beta_1})^2\right)\right|\right).$$

If $\beta_1 \le \beta_2 \le \beta_3$, then we obtain $\mu_{\beta_1} \le \mu_{\beta_2} \le \mu_{\beta_3}$ and $\upsilon_{\beta_1} \ge \upsilon_{\beta_2} \ge \upsilon_{\beta_3}$ following Definition 2.3.

Let $D = d(\beta_1, \beta_3) - d(\beta_1, \beta_2)$, then

$$D = \frac{1}{2}\left((\mu_{\beta_3})^2 - (\nu_{\beta_3})^2 + (\nu_{\beta_2})^2 - (\mu_{\beta_2})^2 + \left|(\mu_{\beta_3})^2 + (\nu_{\beta_3})^2 - \left((\mu_{\beta_1})^2 + (\nu_{\beta_1})^2\right)\right| - \left|(\mu_{\beta_2})^2 + (\nu_{\beta_2})^2 - \left((\mu_{\beta_1})^2 + (\nu_{\beta_1})^2\right)\right|\right).$$

We obtain the following:

(1) if $(\mu_{\beta_1})^2 + (\nu_{\beta_1})^2 \ge \max\{(\mu_{\beta_2})^2 + (\nu_{\beta_2})^2, (\mu_{\beta_3})^2 + (\nu_{\beta_3})^2\}$, then $D = (\nu_{\beta_2})^2 - (\nu_{\beta_3})^2 \ge 0$;
(2) if $(\mu_{\beta_1})^2 + (\nu_{\beta_1})^2 \le \min\{(\mu_{\beta_2})^2 + (\nu_{\beta_2})^2, (\mu_{\beta_3})^2 + (\nu_{\beta_3})^2\}$, then $D = (\mu_{\beta_3})^2 - (\mu_{\beta_2})^2 \ge 0$;
(3) if $(\mu_{\beta_3})^2 + (\nu_{\beta_3})^2 \le (\mu_{\beta_1})^2 + (\nu_{\beta_1})^2 \le (\mu_{\beta_2})^2 + (\nu_{\beta_2})^2$, then $D = (\mu_{\beta_1})^2 + (\nu_{\beta_1})^2 - (\nu_{\beta_3})^2 - (\mu_{\beta_2})^2$;
For (3), because $-(\nu_{\beta_3})^2 \ge -((\mu_{\beta_1})^2 + (\nu_{\beta_1})^2 - (\mu_{\beta_3})^2)$, we have

$$D \ge (\mu_{\beta_1})^2 + (\nu_{\beta_1})^2 - \left((\mu_{\beta_1})^2 + (\nu_{\beta_1})^2 - (\mu_{\beta_3})^2\right) - (\mu_{\beta_2})^2 = (\mu_{\beta_3})^2 - (\mu_{\beta_2})^2 \ge 0.$$

(4) if $(\mu_{\beta_2})^2 + (\nu_{\beta_2})^2 \le (\mu_{\beta_1})^2 + (\nu_{\beta_1})^2 \le (\mu_{\beta_3})^2 + (\nu_{\beta_3})^2$, then $D = (\mu_{\beta_3})^2 + (\nu_{\beta_2})^2 - ((\mu_{\beta_1})^2 + (\nu_{\beta_1})^2)$.

For (4), because $(\mu_{\beta_3})^2 \geq (\mu_{\beta_1})^2 + (\nu_{\beta_1})^2 - (\nu_{\beta_3})^2$, we have

$$D \geq \left((\mu_{\beta_1})^2 + (\nu_{\beta_1})^2 - (\nu_{\beta_3})^2\right) + (\nu_{\beta_2})^2 - \left((\mu_{\beta_1})^2 + (\nu_{\beta_1})^2\right) = (\nu_{\beta_2})^2 - (\nu_{\beta_3})^2 \geq 0.$$

Combing from (1) to (4) yields $d(\beta_1, \beta_3) \geq d(\beta_1, \beta_2)$.
Analogously, we can also prove $d(\beta_1, \beta_3) \geq d(\beta_2, \beta_3)$, which completes the proof of this theorem.

## 4. CONCLUSIONS

In this note, we point out an error to the proof of Theorem 3.4 in the literature[3] by a counterexample. A new proof is therefore provided. As we mentioned in the introduction that the concept of distance measure for PFNs is motived by the Hamming distance of intuitionistic fuzzy numbers provided in Szmidt and Kacprzyk's study, we can accordingly infer and then verify that IFNs also preserve this key property based on the intuitionistic fuzzy Hamming distance. That is to say, for any three IFNs $\alpha_j = I(\mu_{\alpha_j}, \nu_{\alpha_j})(j = 1, 2, 3)$ , if $\alpha_1 \leq \alpha_2 \leq \alpha_3$, then we can prove $d(\alpha_1, \alpha_2) \leq d(\alpha_1, \alpha_3)$ and $d(\alpha_2, \alpha_3) \leq d(\alpha_1, \alpha_3)$ similarly by using our proof of Theorem 3.4 for reference. Considering the further theoretical research and developmental actuality about Theorem 3.4, the revised proof of it provides a sound theoretical foundation.

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## References

1. Yager RR. Pythagorean fuzzy subsets. In: Proc Joint IFSA World Congress and NAFIPS Annual Meeting, Edmonton, Canada; 2013. pp 57–61.
2. Yager RR. Pythagorean membership grades in multi-criteria decision making. IEEE Trans Fuzzy Syst 2014;22:958–965.
3. Zhang X, Xu Z. Extension of TOPSIS to multiple criteria decision making with Pythagorean fuzzy sets. Int J Intell Syst 2014;29(12):1061–1078.
4. Szmidt E, Kacprzyk J. Distances between intuitionistic fuzzy sets. Fuzzy Sets Syst 2000;114(3):505–518.