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The residual extropy of order statistics

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ABSTRACT

Residual extropy was proposed to measure residual uncertainty of a random variable. Monotone properties and characterization results of this measure were studied. Similar properties of the proposed measure of order statistics were also discussed.

1. Introduction

Let X be a non-negative and absolutely continuous random variable with probability density function (pdf) f. To measure the uncertainty contained in X, the entropy was defined by Shannon (1948) as follows,

$$H(X) = -\int_0^\infty f(x)\log f(x)\,\mathrm{d}x.$$

Recently, an alternative measure of uncertainty called extropy was proposed by Lad et al. (2015). For random variable X, its extropy is defined as

$$J(X) = -\frac{1}{2}\int_0^\infty f^2(x)\,\mathrm{d}x. \tag{1.1}$$

One statistical application of extropy is to score the forecasting distributions. For example, under the total log scoring rule, the expected score of a forecasting distribution equals the negative sum of the entropy and extropy of this distribution (Gneiting and Raftery, 2007). In commercial or scientific areas such as astronomical measurements of heat distributions in galaxies, the extropy has been universally investigated (Furuichi and Mitroi, 2012; Vontobel, 2013). Most recently, Qiu (2017) further studied this new measure, exploring some characterization results, monotone properties and lower bounds of extropy of order statistics and record values.

As pointed out by Asadi and Ebrahimi (2000), if X is regarded as the lifetime of a new unit, then $H(X)$ is no longer useful for measuring the uncertainty about remaining lifetime of the unit. In such situations, one should consider the residual entropy

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of X, which was proposed by Ebrahimi (1996) as the entropy of $X_t = [X - t | X \geq t]$, i.e.,

$$H(X_t) = -\int_t^{\infty} \frac{f(x)}{\bar{F}(t)} \log \frac{f(x)}{\bar{F}(t)} \,\mathrm{d}x, \tag{1.2}$$

where $\bar{F}$ is the survival function of X. Analogous to (1.2), the residual extropy of X is defined as the extropy of X_t in this paper. It is shown that the residual extropy of X is determined uniquely by its failure rate function in Section 2. Based on this point, several distributions are characterized in terms of its residual extropy. In Section 3, some monotone properties of residual extropy of the first order statistic are built. We also show that the underlying distributions can be characterized by the residual extropy of order statistics.

2. Residual extropy and characterization results

2.1. Residual extropy

By analogy to Ebrahimi (1996), we propose the following definition of the residual extropy. For random variable X, its residual extropy is defined as

$$J_t(X) \triangleq J(X_t) = -\frac{1}{2}\int_0^{\infty} f_t^2(x)\,\mathrm{d}x = -\frac{1}{2\bar{F}^2(t)}\int_t^{\infty} f^2(x)\,\mathrm{d}x, \quad t \geq 0, \tag{2.1}$$

where $f_t(x) = f(x+t)/\bar{F}(t),\ x \geq 0,\ t \geq 0$, is the pdf of X_t. It is obvious that the residual extropy of a continuous distribution is always negative, while the residual entropy of a continuous distribution may take any value on the extended real line, including $-\infty$ and ∞. It should be noted that if we put $t = 0$ in (2.1), then we get $J_0(X) = J(X)$, which coincides with (1.1).

The next theorem shows that the residual extropy of a random variable is determined uniquely by its failure rate function.

Theorem 2.1. *The residual extropy $J_t(X)$ of X is determined uniquely by $r_X(t)$, where $r_X(t) = f(t)/\bar{F}(t),\ t \geq 0$ is the failure rate function of X.*

Proof. It is obvious from (2.1) that

$$\begin{aligned}\frac{\mathrm{d}J_t(X)}{\mathrm{d}t} &= -\frac{1}{2\bar{F}^4(t)}\left[-f^2(t)\bar{F}^2(t) + 2\bar{F}(t)f(t)\int_t^{\infty} f^2(x)\,\mathrm{d}x\right] \\ &= \frac{1}{2}\left[r_X^2(t) + 4r_X(t)J_t(X)\right].\end{aligned}$$

Thus, we have

$$\frac{\mathrm{d}J_t(X)}{\mathrm{d}t} - 2r_X(t)J_t(X) = \frac{1}{2}r_X^2(t). \tag{2.2}$$

Solving the above differential equation leads to

$$J_t(X) = \mathrm{e}^{2\int r_X(t)\,\mathrm{d}t}\left[\frac{1}{2}\int r_X^2(t)\mathrm{e}^{-2\int r_X(t)\,\mathrm{d}t}\,\mathrm{d}t + C\right], \tag{2.3}$$

where C is a constant and is determined by $J_t(X)|_{t=0} = J(X)$. This completes the proof. □

Remark 2.2. It follows from (2.2) that $J_t(X)$ is increasing (decreasing) in t if and only if $J_t(X) \geq (\leq) - r_X(t)/4$.

Example 2.3. Let X be a Pareto random variable with pdf $f(x) = \gamma\alpha^{\gamma}/(\alpha + x)^{\gamma+1},\ x \geq 0,\ \alpha, \gamma > 0$. Obviously, $r_X(x) = \gamma/(\alpha + x),\ x \geq 0$. It follows from (2.3) that

$$J_t(X) = \mathrm{e}^{2\int \frac{\gamma}{\alpha+t}\,\mathrm{d}t}\left[\frac{1}{2}\int \frac{\gamma^2}{(\alpha+t)^2}\mathrm{e}^{-2\int \frac{\gamma}{\alpha+t}\,\mathrm{d}t}\,\mathrm{d}t + C\right] = -\frac{\gamma^2}{2(2\gamma+1)}\frac{1}{\alpha+t} + C(\alpha+t)^{2\gamma}, \quad t \geq 0.$$

Letting $t = 0$, we have

$$J_t(X)|_{t=0} = -\frac{\gamma^2}{2\alpha(2\gamma+1)} + C\alpha^{2\gamma} = J(X) = -\frac{\gamma^2}{2\alpha(2\gamma+1)}.$$

Thus, $C = 0$ and $J_t(X) = -\gamma^2/[2(2\gamma+1)(\alpha+t)],\ t \geq 0$. Obviously, $J_t(X)$ is increasing in t.

Table 1 lists the residual extropy (entropy) for some commonly-used distributions.

Next we will investigate alternative conditions under which $J_t(X)$ is decreasing in t. To this end, we first recall the definitions of two stochastic orders.

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Table 1
Residual extropy/entropy for some commonly-used distributions.

Name	pdf	Residual extropy	Residual entropy
Finite range	$a(1-x)^{a-1},\ x\in(0,1),\ a>1$	$-\frac{a^2}{(4a-2)(1-t)}$	$\log\frac{1-t}{a}+\frac{a-1}{a}$
Uniform	$\frac{1}{b},\ x\in(0,b),\ b>0$	$-\frac{1}{2(b-t)}$	$\log(b-t)$
Exponential	$\lambda e^{-\lambda x},\ x\geq 0,\ \lambda>0$	$-\frac{\lambda}{4}$	$1-\log\lambda$
Pareto	$\frac{\gamma\alpha^\gamma}{(\alpha+x)^{\gamma+1}},\ x\geq 0,\ \alpha,\gamma>0$	$-\frac{\gamma^2}{2(2\gamma+1)}\frac{1}{\alpha+t}$	$\log\frac{\alpha+t}{\gamma}+\frac{\gamma+1}{\gamma}$
Power	$ax^{a-1},\ x\in(0,1),\ a>0$	$-\frac{a^2(1-t^{2a-1})}{2(2a-1)(1-t^a)^2}$	$\log\frac{1-t^a}{a}+\frac{(a-1)t^a\log t}{1-t^a}+\frac{a-1}{a}$

Definition 2.4 (*Shaked and Shanthikumar, 2007*)**.** Let X and Y be two non-negative random variables with survival functions $\bar{F}$ and $\bar{G}$, pdfs f and g, respectively. X is said to be smaller than Y

(1) in the likelihood ratio order, denoted by $X\leq_{\rm lr}Y$, if $f(x)/g(x)$ is decreasing in $x\geq 0$,
(2) in the usual stochastic order, denoted by $X\leq_{\rm st}Y$, if $\bar{F}(x)\leq\bar{G}(x)$ for all $x\geq 0$.

It is well known that $X\leq_{\rm lr}Y\Longrightarrow X\leq_{\rm st}Y$, and $X\leq_{\rm st}Y$ if and only if $E\phi(X)\leq E\phi(Y)$ for all increasing functions ϕ.

Theorem 2.5. *Let X be a random variable with cumulative distribution function (cdf) F and pdf f. If $f(F^{-1}(x))$ is increasing in $x\geq 0$, then $J_t(X)$ is decreasing in $t\geq 0$.*

Proof. Let U_t be a random variable uniformly distributed on $(F(t),\ 1)$ with pdf $g_t(x)=1/\bar{F}(t),\ F(t)<x<1$. Then, it follows from (2.1) that

$$\begin{aligned}J_t(X)&=-\frac{1}{2\bar{F}^2(t)}\int_{F(t)}^{1}f(F^{-1}(u))\,du\\&=-\frac{1}{2\bar{F}(t)}\int_{F(t)}^{1}g_t(u)f(F^{-1}(u))\,du\\&=-\frac{1}{2\bar{F}(t)}E\left[f(F^{-1}(U_t))\right].\end{aligned}$$

Suppose that $0\leq t_1<t_2$. Then $g_{t_1}(x)/g_{t_2}(x)=\infty$ if $F(t_1)<x\leq F(t_2)$, and $g_{t_1}(x)/g_{t_2}(x)=\bar{F}(t_2)/\bar{F}(t_1)$, a constant, if $F(t_2)<x<1$. Thus, $g_{t_1}(x)/g_{t_2}(x)$ is decreasing in $x\in(F(t_1),\ 1)$, which implies $U_{t_1}\leq_{\rm lr}U_{t_2}$. Hence, $U_{t_1}\leq_{\rm st}U_{t_2}$ and $0\leq E\left[f(F^{-1}(U_{t_1}))\right]\leq E\left[f(F^{-1}(U_{t_2}))\right]$ by the assumption that $f(F^{-1}(x))$ is an increasing function. Since $0<1/\bar{F}(t_1)\leq 1/\bar{F}(t_2)$, we conclude the desired result by noting that

$$J_{t_1}(X)=-\frac{1}{2\bar{F}(t_1)}E\left[f(F^{-1}(U_{t_1}))\right]\geq-\frac{1}{2\bar{F}(t_2)}E\left[f(F^{-1}(U_{t_2}))\right]=J_{t_2}(X).\quad\Box$$

Remark 2.6. The quantity $f(F^{-1}(x))$ is known as the density-quantile function in the literature, and is used to approximate the moments of order statistics (David and Nagaraja, 2003).

Remark 2.7. If X has the finite range distribution as in Table 1, then $f(F^{-1}(x))=a(1-x)^{(a-1)/a}$ is decreasing in $x\in(0,1)$. However, we know from Table 1 that $J_t(X)$ is decreasing in t. So, the condition in Theorem 2.5 that $f(F^{-1}(x))$ is increasing in x is sufficient but not necessary.

Example 2.8. Let X be a B(2, 1) random variable with cdf $F(x)=x^2,\ 0<x<1$, then $f(F^{-1}(x))=2\sqrt{x},\ 0<x<1$. The condition of Theorem 2.8 is satisfied. Thus, the residual extropy of X is decreasing in t.

2.2. Characterizations of several distributions

The pdfs of some commonly-used distributions have been given in Table 1. Next we characterize some distributions in terms of residual extropy.

Theorem 2.9. *Let X be a random variable with failure rate r_X. If for all $t\geq 0$, $J_t(X)=-kr_X(t)$, where k is a non-negative constant. Then X has*

(1) *a finite range distribution if and only if $k>1/4$,*
(2) *an exponential distribution if and only if $k=1/4$,*
(3) *a Pareto distribution if and only if $k<1/4$.*

Proof. The necessities of Parts (1)–(3) can be verified easily by using Table 1. Next we only consider sufficiencies. Suppose that $J_t(X) = -kr_X(t)$ for all $t \geq 0$. It follows from (2.2) that

$$\frac{r'_X(t)}{r^2_X(t)} = -\frac{1-4k}{2k}, \quad t \geq 0.$$

Solving this equation yields $r_X(t) = 1/(pt + d),\ t \geq 0$, where $p = (1 - 4k)/(2k)$, $d = 1/r_X(0)$.

(1) If $k > 1/4$, then $p < 0$, and $r_X(t)$ becomes the failure rate of a finite range distribution.
(2) If $k = 1/4$, then $p = 0$, and $r_X(t)$ turns out to be a constant, which is just the condition under which X has an exponential distribution.
(3) If $k < 1/4$, then $p > 0$, and $r_X(t)$ becomes the failure rate of a Pareto distribution.

This completes the proof by noting that the distribution function is determined uniquely by its failure rate. □

3. Residual extropy of order statistics

Order statistics can be used in many fields, including statistical inference, goodness-of-fit tests, reliability, and quality control. For example, in reliability theory, order statistics are used for statistical modeling. The ith order statistic in a sample of size n represents the life length of an $(n - i + 1)$-out-of-n system. Recently, several authors studied the subject of characterizing underlying distribution of a sample based on the entropy or its generalized versions of order statistics. Baratpour et al. (2007, 2008) showed that Shannon entropy and Rényi entropy of the ith order statistic can characterize the underlying distribution uniquely. Similar results can be found in Baratpour (2010) for cumulative residual entropy of the first order statistic, and in Gupta et al. (2014) for dynamic entropy of the ith order statistic. In this section, we first study the monotone properties of residual extropy, and then we show that the residual extropy of order statistics can also determine the underlying distribution uniquely.

3.1. Monotone properties

Let $X_1, X_2, \ldots, X_n$ be independent random samples of size n from population X with cdf F and pdf f. Denote by $X_{1:n} \leq \cdots \leq X_{n:n}$ the order statistics of $X_1, X_2, \ldots, X_n$. The survival function and pdf of $X_{1:n}$ are given by $\bar{F}_{1:n}(x) = \bar{F}^n(x)$ and $f_{1:n}(x) = n\bar{F}^{n-1}(x)f(x),\ x \geq 0$, respectively. If f is decreasing, then $f_{1:n}(F^{-1}_{1:n}(x)) = nx^{1-1/n}f[F^{-1}(1 - x^{1/n})]$ is increasing in $x \in (0,\ 1)$. Thus, we have the following result by Theorem 2.5.

Theorem 3.1. *If X has a decreasing pdf f on $[0,\ \infty)$, then $J_t(X_{1:n})$ is decreasing in $t \geq 0$.*

The counterexample below shows that the result in Theorem 3.1 could not be generalized to $X_{i:n},\ i > 1$.

Counterexample 3.2. *Let X be a random variable with decreasing pdf $f(x) = 1/(1 + x)^2,\ x \geq 0$. The cdf of $X_{2:2}$ is given by $F_{2:2}(x) = x^2/(1 + x)^2,\ x \geq 0$. Thus,*

$$J_t(X_{2:2}) = -2\frac{1}{\left[1 - \frac{t^2}{(1+t)^2}\right]^2} \int_t^\infty \frac{x^2}{(1+x)^6}\,\mathrm{d}x, \quad t \geq 0.$$

Since $J_{0.4}(X_{2:2}) = -0.0676073 < -0.0624589 = J_{0.8}(X_{2:2})$, $J_t(X_{2:2})$ is not decreasing in t.

The following theorem shows that $J_t(X_{1:n})$ is also decreasing in n if f is decreasing.

Theorem 3.3. *If X has a decreasing pdf f on $[0,\ \infty)$, then $J_t(X_{1:n})$ is decreasing in $n \geq 1$.*

Proof. We note that the pdf of $[X_{1:n}|X_{1:n} > t]$ is given by

$$g^t_{1:n}(x) = \frac{n\bar{F}^{n-1}(x)f(x)}{\bar{F}^n(t)}, \quad x \geq t.$$

We also note that

$$\frac{g^t_{1:(2n-1)}(x)}{g^t_{1:(2n+1)}(x)} = \frac{(2n-1)\bar{F}^2(t)}{(2n+1)\bar{F}^2(x)}$$

is increasing in $x \in [t, \infty)$. Thus, $[X_{1:(2n+1)}|X_{1:(2n+1)} > t] \leq_{\mathrm{lr}} [X_{1:(2n-1)}|X_{1:(2n-1)} > t]$, which implies $[X_{1:(2n+1)}|X_{1:(2n+1)} > t] \leq_{\mathrm{st}} [X_{1:(2n-1)}|X_{1:(2n-1)} > t]$. If f is decreasing, then,

$$\mathrm{E}[f(X_{1:(2n+1)})|X_{1:(2n+1)} > t] \geq \mathrm{E}[f(X_{1:(2n-1)})|X_{1:(2n-1)} > t].$$

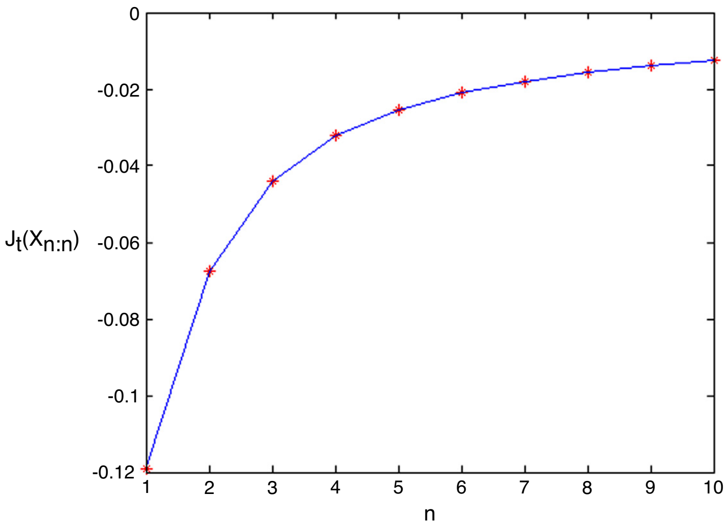

Fig. 1. $J_t(X_{n:n})$ of a random variable with pdf $f(x) = 1/(1+x)^2,\ x \geq 0$ for $t = 0.4$ and $n = 1, 2, \ldots, 10$.

Now, it follows from (2.1) that

$$\begin{aligned} J_t(X_{1:n}) &= -\frac{n^2}{2\bar{F}^{2n}(t)} \int_t^\infty \bar{F}^{2n-2}(x) f^2(x)\,\mathrm{d}x \\ &= -\frac{n^2}{2(2n-1)\bar{F}(t)} \int_t^\infty \frac{(2n-1)\bar{F}^{2n-2}(x)f(x)}{\bar{F}^{2n-1}(t)} f(x)\,\mathrm{d}x \\ &= -\frac{n^2}{2(2n-1)\bar{F}(t)} \int_t^\infty g^t_{1:(2n-1)}(x) f(x)\,\mathrm{d}x \\ &= -\frac{n^2}{2(2n-1)\bar{F}(t)} \mathrm{E}[f(X_{1:(2n-1)})|X_{1:(2n-1)} > t]. \end{aligned} \tag{3.1}$$

Hence,

$$\begin{aligned} \frac{J_t(X_{1:n})}{J_t(X_{1:(n+1)})} &= \frac{n^2(2n+1)}{(n+1)^2(2n-1)} \frac{\mathrm{E}[f(X_{1:(2n-1)})|X_{1:(2n-1)} > t]}{\mathrm{E}[f(X_{1:(2n+1)})|X_{1:(2n+1)} > t]} \\ &\leq \frac{\mathrm{E}[f(X_{1:(2n-1)})|X_{1:(2n-1)} > t]}{\mathrm{E}[f(X_{1:(2n+1)})|X_{1:(2n+1)} > t]} \\ &\leq 1. \end{aligned}$$

This completes the proof since the residual extropy of a random variable is non-positive. □

One may wonder whether the first order statistic $X_{1:n}$ in Theorem 3.3 can be replaced by order statistics $X_{i:n},\ i > 1$. The following counterexample gives a negative answer.

Counterexample 3.4. *Let X be as defined in Counterexample 3.2. The residual extropy of $X_{n:n}$ is given by*

$$J_t(X_{n:n}) = -\frac{1}{2\left[1 - \frac{t^n}{(1+t)^{2n}}\right]^2} \int_t^\infty \frac{n^2 x^{2n-2}}{(1+x)^{2n+2}}\,\mathrm{d}x, \quad t \geq 0.$$

Fig. 1 shows that $J_{0.4}(X_{n:n})$ is increasing in $n \in \{1, 2, \ldots, 10\}$ although f is decreasing.

3.2. Characterizations of underlying distributions of order statistics

Qiu (2017) showed that the underlying distribution can be characterized uniquely by the extropy of the ith order statistic. In this subsection, we show that the residual extropy of the ith order statistic can also characterize the underlying distribution uniquely.

Theorem 3.5. *Let $J_t(X_{1:n})$ and $J_t(Y_{1:n})$ be the residual extropy of the first order statistic from X and Y, respectively. Then $X \stackrel{d}{=} Y$ if and only if for all $t \geq 0$ and $n \geq 1$, $J_t(X_{1:n}) = J_t(Y_{1:n})$.*

Proof. It suffices to prove the sufficiency. Denote the residual lifetime of $X_{1:n}$ at age $t \geq 0$ by $X_{1:n;\,t} = [X_{1:n} - t | X_{1:n} \geq t]$. Then the survival function of $X_{1:n;\,t}$ is given by $\bar{F}_{1:n;\,t}(x) = \bar{F}_{1:n}(t+x)/\bar{F}_{1:n}(t) = \left[\bar{F}(t+x)/\bar{F}(t)\right]^n$, $x \geq 0$, $t \geq 0$. Therefore,

$$X_{1:n;\,t} \stackrel{d}{=} \min\{X_{1;\,t}, X_{2;\,t}, \ldots, X_{n;\,t}\}, \tag{3.2}$$

where $X_{i;\,t} = [X_i - t | X_i \geq t]$, $i = 1, 2, \ldots, n$. If $J_t(X_{1:n}) = J_t(Y_{1:n})$ for all $t \geq 0$ and $n \geq 1$, then by (3.2) and Remark 3.3 in Qiu (2017) that $X_t \stackrel{d}{=} Y_t$ for all $t \geq 0$. i.e. $F_t(x) = G_t(x)$ for all $x \geq 0$ and $t \geq 0$, where $F_t(x)$ and $G_t(x)$ are cdfs of X_t and Y_t, respectively. Thus, $F(x+t) = \bar{F}(t)G(x+t)/\bar{G}(t)$ for all $x \geq 0$ and $t \geq 0$. Letting $x \to \infty$ yields that $\bar{F}(t) = \bar{G}(t)$ for all $t \geq 0$, that is, X and Y have the same distribution function. □

To generalize Theorem 3.5 from $X_{1:n}$ to $X_{i:n}$, $i \geq 1$, we consider the problem of finding a sufficient condition for the unique solution of the initial value problem (IVP)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y), \quad y(x_0) = y_0, \tag{3.3}$$

where f is a function of two variables whose domain is a region $D \subset \mathbb{R}^2$, (x_0, y_0) is a point in D and y is the unknown function. By the solution of (3.3), we find a function φ which satisfies the following conditions: (i) φ is differentiable on I, (ii) the growth of φ lies in D, (iii) $\varphi(x_0) = y_0$ and (iv) $\varphi'(x) = f(x, \varphi(x))$, for all $x \in I$. The following theorem will be used to prove our characterization results.

Theorem 3.6 (Gupta and Kirmani, 1998)**.** *Let f be a continuous function defined in a domain $D \subset \mathbb{R}^2$ and let f satisfy Lipschitz condition (with respect to y) in D, that is $|f(x, y_1) - f(x, y_2)| \leq k|y_1 - y_2|$, $k > 0$, for every point (x, y_1) and (x, y_2) in D. Then the function $y = \varphi(x)$ satisfying the IVP $y' = f(x, y)$ and $y(x_0) = y_0$, $x \in I$, is unique.*

We use the following lemma to present a sufficient condition which guarantees that the Lipschitz condition is satisfied in D.

Lemma 3.7 (Gupta and Kirmani, 2008)**.** *Suppose that the function f is continuous in a convex region $D \subset \mathbb{R}^2$, $\frac{\partial f}{\partial y}$ exists and is continuous in D. Then f satisfies the Lipschitz condition in D.*

Theorem 3.8. *Let $J_t(X_{i:n})$ and $J_t(Y_{i:n})$ be the residual extropy of the ith order statistic from X and Y, respectively. Then $X \stackrel{d}{=} Y$ if and only if for all $t \geq 0$ and $n \geq i$, $J_t(X_{i:n}) = J_t(Y_{i:n})$.*

Proof. It suffices to prove the sufficiency. It is known from (2.2) that

$$\frac{\mathrm{d}J_t(X_{i:n})}{\mathrm{d}t} - 2r_{X_{i:n}}(t)J_t(X_{i:n}) = \frac{1}{2}r^2_{X_{i:n}}(t), \quad t \geq 0.$$

Taking derivative of the above equation with respect to t, we have

$$\frac{\mathrm{d}r_{X_{i:n}}(t)}{\mathrm{d}t} = \frac{\frac{\mathrm{d}^2}{\mathrm{d}t^2}J_t(X_{i:n}) - 2r_{X_{i:n}}(t)\frac{\mathrm{d}}{\mathrm{d}t}J_t(X_{i:n})}{r_{X_{i:n}}(t) + 2J_t(X_{i:n})}.$$

Assume that $J_t(X_{i:n}) = J_t(Y_{i:n}) = v(t)$ for all $t \geq 0$, $n \geq i$. Then for all $t \geq 0$,

$$\frac{\mathrm{d}r_{X_{i:n}}(t)}{\mathrm{d}t} = \psi(t, r_{X_{i:n}}(t)), \quad \frac{\mathrm{d}r_{Y_{i:n}}(t)}{\mathrm{d}t} = \psi(t, r_{Y_{i:n}}(t)),$$

where

$$\psi(t, y) = \frac{v''(t) - 2yv'(t)}{y + 2v(t)}.$$

It follows from Theorem 3.6 and Lemma 3.7 that $r_{X_{i:n}}(t) = r_{Y_{i:n}}(t)$ for all $t \geq 0$, which implies $F_{i:n}(t) = G_{i:n}(t)$ for all $t \geq 0$, where $F_{i:n}(t)$ and $G_{i:n}(t)$ are cdfs of $X_{i:n}$ and $Y_{i:n}$, respectively. In view of $F(t) = B^{-1}_{i,n-i+1}(F_{i:n}(t))$, $G(t) = B^{-1}_{i,n-i+1}(G_{i:n}(t))$, $t \geq 0$, where $B_{i,n-i+1}(t)$ is the cdf of the beta distribution with parameters i and $(n-i+1)$, we have $F(t) = G(t)$ for all $t \geq 0$. The desired result is proved. □

To end this subsection, we give new characterizations of several distributions in terms of residual extropy of the first order statistic. The proof is similar to that of Theorem 2.9 and hence, is omitted by noting that $r_{X_{1:n}}(t) = nr_X(t)$ for all $t \geq 0$.

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Theorem 3.9. *Let X be a random variable with failure rate r_X. If $J_t(X_{1:n}) = -kr_X(t)$ for all $t \geq 0$, where k is a non-negative constant. Then X has*

(1) *a finite range distribution if and only if $k > n/4$,*
(2) *an exponential distribution if and only if $k = n/4$,*
(3) *a Pareto distribution if and only if $k < n/4$.*

3.3. Two corollaries and further discussions

Given the $(i-1)$th order statistic, the ith order statistic can be viewed as the first order statistic from a residual distribution. Therefore, we have the following two corollaries.

Corollary 3.10. *If X has a decreasing pdf f on $[0, \infty)$, then*

(1) *$J(X_{i:n} - X_{(i-1):n}|X_{(i-1):n} = t)$ is decreasing in $t \geq 0$ for fixed n and $i \geq 2$,*
(2) *$J(X_{i:n} - X_{(i-1):n}|X_{(i-1):n} = t)$ is decreasing in n for fixed $t \geq 0$ and $i \geq 2$.*

Proof. By Proposition 2.1 in Hu and Zhuang (2005), we have $[X_{i:n} - X_{(i-1):n}|X_{(i-1):n} = t] \stackrel{d}{=} X^t_{1:(n-i+1)}$, where $X^t_{1:(n-i+1)}$ is the first order statistic in a sample of size $(n-i+1)$ from distribution $F_t(x) = 1 - \bar{F}(t+x)/\bar{F}(t)$, $x \geq 0, t \geq 0$. According to (3.2), we further have

$$X^t_{1:(n-i+1)} \stackrel{d}{=} \min\{X_{1;\,t}, X_{2;\,t}, \ldots, X_{(n-i+1);\,t}\} \stackrel{d}{=} X_{1:(n-i+1);\,t}.$$

Thus, $J(X_{i:n} - X_{(i-1):n}|X_{(i-1):n} = t) = J(X_{1:(n-i+1);\,t}) = J_t(X_{1:(n-i+1)})$ and the proof is completed by Theorems 3.1 and 3.3. □

Similarly, we can prove the following corollary of Theorem 3.5.

Corollary 3.11. *$X \stackrel{d}{=} Y$ if and only if for all $t \geq 0$ and $n \geq i \geq 2$, $J(X_{i:n} - X_{(i-1):n}|X_{(i-1):n} = t) = J(Y_{i:n} - Y_{(i-1):n}|Y_{(i-1):n} = t)$.*

In addition, the characterization results in this paper may be used in goodness-of-fit tests. Let $X_1, \ldots, X_n$ be random samples of size n, from a population with unknown cdf F. We wish to test the null hypothesis $H_0 : F(x) = F_0(x)$ for all $x \geq 0$, against $H_1 : F(x) \neq F_0(x)$ for some $x \geq 0$, where $F_0(x) = 1 - e^{-\lambda x}$, $\lambda > 0, x \geq 0$. The importance of this test is that the exponential distribution plays an important role in reliability theory. Thus, according to Theorem 3.9, the above goodness-of-fit problem is equivalent to test the null hypothesis $H_0 : J_t(X_{1:n}) + n\lambda/4 = 0$ for all $t \geq 0$, against $H_1 : J_t(X_{1:n}) + n\lambda/4 \neq 0$ for some $t \geq 0$. Recall that the maximum likelihood estimate for parameter λ is $n/\sum_{i=1}^n X_i$. If we have a good estimate for $J_t(X_{1:n})$, say $\hat{J}_t(X_{1:n})$, then the large values of $\hat{J}_t(X_{1:n}) + n^2/(4\sum_{i=1}^n X_i)$ can be regarded as a symptom of non-exponentiality and therefore we reject the null hypothesis.

Note from (3.1) that $J_t(X_{1:n})$ can be rewritten as

$$J_t(X_{1:n}) = -\frac{n^2}{2\bar{F}^{2n}(t)} \int_0^1 (1-u)^{2n-2} \left[\frac{\mathrm{d}F^{-1}(u)}{\mathrm{d}u}\right]^{-1} 1(u \geq F(t))\,\mathrm{d}u,$$

where $1(u \geq F(t))$ denotes the indicator function of $\{u \geq F(t)\}$. Similar to Park (1999), a sample estimate of $J_t(X_{1:n})$, based on sample of size k, may be constructed as

$$\hat{J}_t(X_{1:n}) = -\frac{n^2}{2[1-\hat{F}(t)]^{2n}} \sum_{i=1}^{k} \left[\left(1 - \frac{i}{k+1}\right)^{2n-2} \frac{2m}{k^2(X_{i+m:k} - X_{i-m:k})} 1\left(\frac{i}{k+1} \geq \hat{F}(t)\right)\right],$$

where $m \leq k/2$ is a positive integer which is called window size, $\hat{F}$ is the empirical distribution function of X, $X_{i:k} = X_{1:k}$ if $i < 1$ and $X_{i:k} = X_{k:k}$ if $i > k$.

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