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Variational iteration method – a kind of non-linear analytical technique: some examples

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Abstract

In this paper, a new kind of analytical technique for a non-linear problem called the variational iteration method is described and used to give approximate solutions for some well-known non-linear problems. In this method, the problems are initially approximated with possible unknowns. Then a correction functional is constructed by a general Lagrange multiplier, which can be identified optimally via the variational theory. Being different from the other non-linear analytical methods, such as perturbation methods, this method does not depend on small parameters, such that it can find wide application in non-linear problems without linearization or small perturbations. Comparison with Adomian's decomposition method reveals that the approximate solutions obtained by the proposed method converge to its exact solution faster than those of Adomian's method.

Keywords: Variational iteration method; Duffing equation; Non-linear equations

1. Introduction

In 1978, Inokuti et al. [1] proposed a general Lagrange multiplier method to solve non-linear problems, which was first proposed to solve problems in quantum mechanics (see Ref. [1] and the references cited therein). The main feature of the method is as follows: the solution of a mathematical problem with linearization assumption is used as initial approximation or trial-function, then a more highly precise approximation at some special point can be obtained. Considering the following general non-linear system.

$$Lu + Nu = g(x), \tag{1}$$

where L is a linear operator, and N is a non-linear operator.

Assuming $u_0(x)$ is the solution of $Lu = 0$, according to Ref. [1], we can write down an expression to correct the value of some special point, for example at $x = 1$

$$u_{\mathrm{cor}}(1) = u_0(1) + \int_0^1 \lambda(Lu_0 + Nu_0 - g)\,\mathrm{d}x, \tag{2}$$

where λ is a general Lagrange multiplier [1], which can be identified optimally via variational theory

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[1–3], the second term on the right is called the correction.

The author has modified the above method into an iteration method [4–8] in the following way:

$$u_{n+1}(x_0) = u_n(x_0) + \int_0^{x_0} \lambda(Lu_n + N\tilde{u}_n - g)\,\mathrm{d}x \quad (3)$$

with $u_0(x)$ as initial approximation with possible unknowns, and $\tilde{u}_n$ is considered as a restricted variation [3], i.e. $\delta\tilde{u}_n = 0$. For arbitrary of x_0, we can rewrite Eq. (3) as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^{x} \lambda\{Lu_n(\xi) + N\tilde{u}_n(\xi) - g(\xi)\}\,\mathrm{d}\xi \quad (4)$$

Eq. (4) is called a correction functional. The modified method, or variational iteration method has been shown [4–8] to solve effectively, easily, and accurately a large class of non-linear problems with approximations converging rapidly to accurate solutions. For linear problems, its exact solution can be obtained by only one iteration step due to the fact that the Lagrange multiplier can be exactly identified. Considering the following example:

$$y'' + \omega^2 y = f(t), \text{ with } f(t) = A\sin\omega t + B\sin t. \quad (5)$$

Its correction functional can be written down as follows:

$$y_{n+1}(t) = y_n(t) + \int_0^{t} \lambda\{y_n''(\tau) + \omega^2 y_n(\tau) - f(\tau)\}\,\mathrm{d}\tau. \quad (6)$$

Making the above correction functional stationary, and noticing that $\delta y(0) = 0$

$$\begin{aligned}\delta y_{n+1}(t) &= \delta y_n(t) + \delta\int_0^{t} \lambda\{y_n''(\tau) + \omega^2 y_n(\tau) - f(\tau)\}\mathrm{d}\tau\\ &= \delta y_n(t) + \lambda(\tau)\delta y_n'(\tau)|_{\tau=t} - \lambda'(\tau)\delta y_n(\tau)|_{\tau=t}\\ &\quad + \int_0^{t} (\lambda'' + \omega^2\lambda)\delta y_n|\mathrm{d}\tau = 0\end{aligned}$$

yields the following stationary conditions:

$$\begin{aligned}&\delta y_n{:}\lambda''(\tau) + \omega^2\lambda(\tau) = 0,\\ &\delta y_n'{:}\lambda(\tau)|_{\tau=t} = 0,\\ &\delta y_n{:}1 - \lambda'(\tau)|_{\tau=t} = 0. \quad (7)\end{aligned}$$

The Lagrange multiplier, therefore, can be readily identified,

$$\lambda = \frac{1}{\omega}\sin\omega(\tau - t) \quad (8)$$

as a result, we obtain the following iteration formula

$$y_{n+1}(t) = y_n(t) + \frac{1}{\omega}\int_0^{t} \sin\omega(\tau - t) \times\{y_n''(\tau) + \omega^2 y_n(\tau) - f(\tau)\}\,\mathrm{d}\tau. \quad (9)$$

If we use its complementary solution $y_0 = C_1\cos\omega t + C_2\sin\omega t$ as initial approximation, by the iteration formula (9), we get

$$\begin{aligned}y_1(t) &= y_0(t) + \frac{1}{\omega}\int_0^{t}\sin\omega(\tau - t) \times\{-A\sin\omega\tau - B\sin\tau\}\,\mathrm{d}\tau\\ &= C_1\cos\omega t + C_2\sin\omega t - \frac{A}{2\omega}t\cos\omega t\\ &\quad + \frac{B}{\omega^2 - 1}(\sin t + \sin\omega t) \quad (10)\end{aligned}$$

which is the general solution Eq. (5).

However, if we apply restricted variations to the correction function (6), then its exact solution can be arrived at only by successive iterations. Considering homogenous Eq. (5), i.e. $f(x) = 0$, we re-write the correction functional of Eq. (6) as follows:

$$y_{n+1}(t) = y_n(t) + \int_0^{t} \lambda\{y_n''(\tau) + \omega^2\tilde{y}_n(\tau)\}\,\mathrm{d}\tau, \quad (11)$$

herein $\tilde{y}_n$ is considered a restricted variation, under this condition, the stationary conditions of the above correction functional (11) can be expressed as follows: (noticing that $\delta\tilde{y}_n = 0$)

$$\begin{aligned}&\lambda''(\tau) = 0,\\ &\lambda(\tau)_{\tau=t} = 0,\\ &1 - \lambda'(\tau)|_{\tau=t} = 0. \quad (12)\end{aligned}$$

The Lagrange multiplier, therefore, can be easily identified as follows:

$$\lambda = \tau - t, \quad (13)$$

leading to the following iteration formula

$$y_{n+1}(t) = y_n(t) + \int_0^t (\tau - t)\{y_n''(\tau) + \omega^2 y_n(\tau)\}\, d\tau. \tag{14}$$

If, for example, the initial conditions are $y(0) = 1$ and $y'(0) = 0$, we began with $y_0 = y(0) = 1$, by the above iteration formula (14) we have the following approximate solutions:

$$y_1(t) = 1 + \omega^2 \int_0^t (\tau - t)\, d\tau = 1 - \frac{1}{2!}\omega^2 t^2, \tag{15a}$$

$$y_2(t) = 1 - \frac{1}{2!}\omega^2 t^2 + \int_0^t (\tau - t)\left\{-\omega^2 + \omega^2 - \frac{1}{2!}\omega^4 \tau^2\right\} d\tau,$$

$$= 1 - \frac{1}{2!}\omega^2 t^2 + \frac{1}{4!}\omega^4 t^4, \tag{15b}$$

$$y_n(t) = 1 - \frac{1}{2!}\omega^2 t^2 + \frac{1}{4!}\omega^4 t^s + \cdots + (-1)^n \frac{1}{(2n)!}\omega^{2n} t^{2n}. \tag{15c}$$

Thus we have

$$\lim_{n \to \infty} y_n(t) = \cos \omega t \tag{15d}$$

which is the exact solution.

From the above solution process, we can see clearly that the approximate solutions converge to its exact solution relatively slowly due to the approximate identification of the multiplier. It should be specially pointed out that the more accurate the identification of the multiplier, the faster the approximations converge to its exact solution. The approximately identified multiplier (13) is actually the first-order approximation of its exact one (8), to get a closer approximation than Eq. (13), we expand Eq. (8) as

$$\lambda = \frac{1}{\omega} \sin \omega(\tau - t) \approx \tau - t - \frac{1}{3!}\omega^2 (\tau - t)^3. \tag{16}$$

Substitution of Eq. (16) in Eq. (11) results in the following iteration formula:

$$y_{n+1}(t) = y_n(t) + \int_0^t \left\{\tau - t - \frac{1}{3!}\omega^2(\tau - t)^3\right\} \times \{y_n''(\tau) + \omega^2 y_n(\tau)\}\, d\tau. \tag{17}$$

We also begin with $y_0(t) = 1$, by the same manipulation, we have

$$y_1(t) = 1 + \int_0^t \left\{\tau - t - \frac{1}{3!}\omega^2(\tau - t)^3\right\}\omega^2\, d\tau = 1 - \frac{1}{2!}\omega^2 t^2 + \frac{1}{4!}\omega^4 t^4, \tag{18a}$$

$$y_2(t) = y_1(t) + \int_0^t \{\tau - t - \tfrac{1}{3}\omega^2(\tau - t)^3\} \times \left\{\frac{1}{4!}\omega^6 \tau^4\right\} d\tau$$

$$= 1 - \frac{1}{2!}\omega^2 t^2 + \frac{1}{4!}\omega^4 t^4 - \frac{1}{6!}\omega^6 t^6 + \frac{1}{8!}\omega^8 t^8. \tag{18b}$$

So, it can be seen clearly that the approximations obtained from Eq. (17) converge to its exact solution faster than those obtained from the iteration formula (14).

For non-linear problems, in order to determine the Lagrange multiplier in as simple a manner as possible, the non-linear terms have to be considered as restricted variations, so the above discussed case also applies to the non-linear problems, details are discussed below.

2. Some examples

Now we apply the proposed technique to solve some non-linear examples.

2.1. Example 1. Duffing equation

Duffing equation is widely used by many perturbation techniques to verify their effectiveness, herein we will also apply the Duffing equation with non-linearity of fifth order to illustrate the general evaluation process of the proposed method.

$$\frac{d^2u}{dt^2} + u + \varepsilon u^5 = 0,$$

$$u(0) = A, \quad u'(0) = 0, \tag{19}$$

its correction functional can be written down as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda \left\{ \frac{d^2 u_n(\tau)}{d\tau^2} + u_n(\tau) + \varepsilon \tilde{u}_n^5(\tau) \right\} d\tau, \tag{20}$$

where $\tilde{u}_n$ is considered as a restricted variation.

Its stationary conditions can be obtained as follows:

$$\lambda''(\tau) + \lambda(\tau) = 0,$$

$$\lambda(\tau)|_{\tau=t} = 0,$$

$$1 - \lambda'(\tau)|_{\tau=t} = 0. \tag{21}$$

The multiplier, therefore, can be identified as $\lambda = \sin(\tau - t)$, and the following variational iteration formula can be obtained:

$$u_{n+1}(t) = u_n(t) + \int_0^t \sin(\tau - t) \times \left\{ \frac{d^2 u_n(\tau)}{d\tau^2} + u_n(\tau) + \varepsilon u_n^5(\tau) \right\} d\tau. \tag{22}$$

Assuming that its approximate solution has the form

$$u_0(t) = A \cos \alpha t, \tag{23}$$

where $\alpha(\varepsilon)$ is a non-zero unknown function of ε with $\alpha(0) = 1$.

The substitution of Eq. (23) in Eq. (19) results in the following residual

$$R_0(t) = (-\alpha^2 + 1 + \tfrac{5}{8}\varepsilon A^4) A \cos \alpha t + \tfrac{1}{16}\varepsilon A^5 (\cos 5\alpha t + 5\cos 3\alpha t). \tag{24}$$

By the variational iteration formula (22), we have

$$\begin{aligned} u_1(t) &= A\cos\alpha t + \int_0^t \sin(\tau - t) R_0(\tau)\, d\tau \\ &= A\cos\alpha t + (-\alpha^2 + 1 + \tfrac{5}{8}\varepsilon A^4) \times \frac{A}{\alpha^2 - 1}(\cos\alpha t - \cos t) \\ &\quad + \frac{\varepsilon A^5}{16(25\alpha^2 - 1)}(\cos 5\alpha t - \cos t) \\ &\quad + \frac{5\varepsilon A^5}{16(9\alpha^2 - 1)}(\cos 3\alpha t - \cos t). \end{aligned} \tag{25}$$

In order to ensure that no secular terms appear in the next iteration, resonance must be avoided. To do so, we equate the coefficient of $\cos t$ equal to zero

$$-\frac{\varepsilon A^5}{16(25\alpha^2 - 1)} - \frac{5\varepsilon A^5}{16(9\alpha^2 - 1)} - (-\alpha^2 + 1 + \tfrac{5}{8}\varepsilon A^4)\frac{A}{\alpha^2 - 1} = 0. \tag{26}$$

Thus we have its first-order approximation

$$u_1(t) = \frac{\varepsilon A^5}{16(25\alpha^2 - 1)}\cos 5\alpha t + \frac{5\varepsilon A^5}{16(9\alpha^2 - 1)}\cos 3\alpha t + \frac{5\varepsilon A^5}{8(\alpha^2 - 1)}\cos\alpha t, \tag{27}$$

with α determined from Eq. (26), which can be approximately expressed as

$$\alpha = \sqrt{1 + \tfrac{5}{8}\varepsilon A^4 + O(\varepsilon^2 A^8)}. \tag{28}$$

The function α can also be identified by various methods such as method of weighted residuals (least square method, method of collocation, Galerkin method). Here we will also give a very simple but heuristic technique to determine the unknown function α. Generally speaking, the residual Eq. (24) is not equal to zero, the right-hand side of Eq. (24) would have to vanish if $u_0(t)$ were to be a solution of Eq. (19). We may, however, at least assure the vanishing of the factor of $\cos\alpha t$ by setting

$$\alpha = \sqrt{1 + 5\varepsilon A^4/8}, \tag{29}$$

then by the iteration formula (22), we obtain

$$\bar{u}_1(t) = A\cos\alpha t + \int_0^t \sin(\tau - t) \times\left[\frac{\varepsilon A^5}{16}\cos 5\alpha\tau + \frac{5\varepsilon A^5}{16}\cos 3\alpha\tau\right]d\tau$$
$$= A\cos\alpha t + \frac{\varepsilon A^5}{16(25\alpha^2 - 1)}(\cos 5\alpha t - \cos t) + \frac{5\varepsilon A^5}{16(9\alpha^2 - 1)}(\cos 3\alpha t - \cos t) \tag{30}$$

with α defined as in Eq. (29).

Its period can be calculated as follows:

$$T = \frac{2\pi}{\sqrt{1 + 5\varepsilon A^4/8}}, \tag{31}$$

while the period obtained by perturbation method [9] reads

$$T = 2\pi(1 - 5\varepsilon A^4/16) \tag{32}$$

and the exact one can be readily obtained as follows:

$$T_{ex} = \frac{4}{\sqrt{1 + \frac{1}{3}\varepsilon A^4}}\int_0^{\pi/2}\frac{dx}{\sqrt{1 + k\cos^2 x + k\cos^4 x}} \tag{33}$$

with $k = \frac{1}{3}\varepsilon A^4/(1 + \frac{1}{3}\varepsilon A^4)$.

It should be specially pointed out that the perturbation formula (32) is valid only for small parameter ε, whereas Eq. (31) is valid not only for small parameter, but also for very large parameter, even in the case $\varepsilon A^4 \to \infty$, we have

$$\lim_{\varepsilon A^4\to\infty}\frac{T_{ex}}{T} = \frac{2\sqrt{\frac{15}{8}}}{\pi}\int_0^{\pi/2}\frac{dx}{\sqrt{1 + \cos^2 x + \cos^4 x}} = \frac{2\sqrt{\frac{15}{8}}}{\pi}\times 1.14811 = 1.0008.$$

Therefore, for any value of ε, it can be easily proved that $0 \leqslant |(T_{ex} - T)|/T_{ex} \leqslant 0.08\%$ so the approximate solution obtained by the proposed method is uniformly valid for any value of ε!

2.2. Example 2. Mathematical pendulum [10, 11]

Many of the mathematical methods employed in non-linear oscillators may be successfully tested on one of the simplest mathematical systems: the mathematical pendulum. When friction is neglected, the differential equation governing the free oscillation of the mathematical pendulum is given by

$$u'' + \omega^2\sin u = 0, \tag{34}$$

with initial conditions $u(0) = A$ and $u'(0) = 0$.

In order to apply the variational iteration method to solve the above problem, the approximation $\sin u \approx u - \frac{1}{6}u^3 + \frac{1}{120}u^5$ is used, as a result, we obtain the following correction functional

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda\left\{\frac{d^2u_n(\tau)}{d\tau^2} + [u_n(\tau) - \tfrac{1}{6}\tilde{u}_n^3 + \tfrac{1}{120}\tilde{u}_n^5]\right\}d\tau, \tag{35}$$

where $\tilde{u}_n$ is a restricted variation.

The multiplier can be identified as $\lambda = \sin\omega(\tau - t)/\omega$, leading to the following iteration formula

$$u_{n+1}(t) = u_n(t) + \frac{1}{\omega}\int_0^t \sin\omega(\tau - t)\left\{\frac{d^2u_n(\tau)}{d\tau^2} + \omega^2[u_n(\tau) - \tfrac{1}{6}u_n^3 + \tfrac{1}{120}u_n^5]\right\}d\tau. \tag{36}$$

We begin with the initial approximation $u_0(t) = A\cos\alpha\omega t$, where α is an unknown constant. Substituting the initial approximation into Eq. (34) results in the following residual

$$R_0(t) \approx u_0'' + \omega^2[u_0 - \tfrac{1}{6}u_0^3 + \tfrac{1}{120}u_0^5]$$
$$= A(-\alpha^2 + 1)\omega^2\cos\alpha\omega t - \tfrac{1}{6}\omega^2A^3\cos^3\alpha\omega t + \tfrac{1}{120}\omega^2A^5\cos^5\alpha\omega t$$
$$= A(-\alpha^2 + 1 - \tfrac{1}{8}A^2 + \tfrac{1}{192}A^4)\omega^2\cos\alpha\omega t - (\tfrac{1}{24} - \tfrac{1}{384}A^2)\omega^2A^3\cos 3\alpha\omega t + \tfrac{1}{1920}\omega^2A^5\cos 5\alpha\omega t.$$

In order to solve its first-order approximation in as simple a manner as possible, we equate the coefficient of $\cos\alpha\omega t$ equal to zero by setting

$$\alpha = \sqrt{1 - \tfrac{1}{8}A^2 + \tfrac{1}{192}A^4}. \tag{37}$$

Therefore, the iteration formula (36) yields the following result:

$$u_1(t) = A\cos\alpha\omega t - \frac{(\frac{1}{24} - \frac{1}{384}A^2)A^3}{(9\alpha^2-1)\omega^2}\times(\cos 3\alpha\omega t - \cos\omega t) + \frac{A^5}{1920(25\alpha^2-1)\omega^2}\times(\cos 5\alpha\omega t - \cos\omega t), \quad (38)$$

with α defined as Eq. (37)

Its period can be expressed as follows:

$$T = \frac{2\pi}{\omega\sqrt{1 - \frac{1}{8}A^2 + \frac{1}{192}A^4}}. \quad (39)$$

The approximate period (39) is of high accuracy, for example, when $A = \pi/2$, the value obtained from Eq. (39) is $T = 1.17T_0$, while the exact one is $T_{ex} = 1.16T_0$, where $T_0 = 2\pi/\omega$.

2.3. Example 3. Vibrations of the eardrum [10]

As a third example, we consider the equation of the motion of the human eardrum [10]

$$u'' - \omega^2 u + \varepsilon u^2 = 0,$$
$$u(0) = A, \quad u'(0) = 0. \quad (40)$$

Its correction functional can be written down as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda\left\{\frac{d^2u_n(\tau)}{d\tau^2} + \omega^2 u_n + \varepsilon\tilde{u}_n^2\right\}d\tau, \quad (41)$$

where $\tilde{u}_n$ is a restricted variation.

The Lagrange multiplier can be readily identified as $\lambda = \sin\omega(\tau - t)/\omega$, and the following iteration formula can be obtained

$$u_{n+1}(t) = u_n(t) + \frac{1}{\omega}\int_0^t \sin\omega(\tau - t)\times\left\{\frac{d^2u_n(\tau)}{d\tau^2} + \omega^2 u_n + \varepsilon u_n^2\right\}d\tau \quad (42)$$

Assuming that the trial-function has the form $u_0(t) = A\cos\alpha\omega t$, where $\alpha(\varepsilon)$ is a non-zero constant with $\alpha(0) = 1$. In view of Eq. (42), we have

$$\begin{aligned} u_1(t) &= A\cos\alpha\omega t + \frac{1}{\omega}\int_0^t \sin\omega(\tau - t)\times\left[(1-\alpha^2)A\omega^2\cos\alpha\omega\tau + \frac{\varepsilon A^2}{2}(1+\cos 2\alpha\omega\tau)\right]d\tau \\ &= A\cos\alpha\omega t - A(\cos\alpha\omega t - \cos\omega t) - \frac{\varepsilon A^2}{2\omega^2}(1-\cos\omega t) + \frac{\varepsilon A^2}{2\omega^2(4\alpha^2-1)}\times(\cos 2\alpha\omega t - \cos\omega t) \\ &= a\cos\omega t - b + c\cos 2\alpha\omega t, \end{aligned} \quad (43)$$

with the definition

$$a = A + \frac{\varepsilon A^2}{2\omega^2} - \frac{\varepsilon A^2}{2\omega^2(4\alpha^2-1)},$$
$$b = \frac{\varepsilon A^2}{2\omega^2}, \quad c = \frac{\varepsilon A^2}{2\omega^2(4\alpha^2-1)}. \quad (44)$$

For small ε, the unknown constant α can be approximately chosen as follows:

$$\alpha = 1 + O(\varepsilon). \quad (45)$$

From Eq. (43) we have

$$u_1 = A\cos\omega t + \frac{\varepsilon A^2}{6\omega^2}(-3 + 2\cos\omega t + \cos 2\omega t) + O(\varepsilon^2) \quad (46)$$

which as the same as that obtained by perturbation method [10], and is of a relatively poor approximation. To obtain an approximation with more high accuracy, we should identify the unknown α such that in the next iteration secular terms will not occur, so the coefficient of $\cos\omega t$ in Eq. (43) must vanish, i.e.

$$a = 0 \ \text{ or } \ \alpha = \tfrac{1}{2}\sqrt{\varepsilon A/(\varepsilon A + 2\omega^2) + 1}. \quad (47)$$

We, therefore, obtain its first-order approximation

$$u_1(t) = -\frac{\varepsilon A^2}{2\omega^2} + \frac{\varepsilon A^2}{2\omega^2(4\alpha^2 - 1)}\cos 2\alpha\omega t \tag{48}$$

with α defined as Eq. (47).

By the same manipulation and by Eq. (42) we obtain its second approximation

$$\begin{aligned} u_2(t) = u_1(t) - &\left(-b\omega^2 + \varepsilon b^2 + \frac{\varepsilon c^2}{2}\right) \\ &\times \frac{1}{\omega^2}(1 - \cos\omega t) \\ &+ (-4\alpha^2\omega^2 c + \omega^2 c - 2\varepsilon bc) \\ &\times \frac{1}{\omega^2(4\alpha^2 - 1)}(\cos 2\alpha\omega t - \cos\omega t) \\ &+ \frac{\varepsilon c^2}{2}\frac{1}{\omega^2(16\alpha^2 - 1)}(\cos 4\alpha\omega t - \cos\omega t), \end{aligned} \tag{49}$$

with b and c defined as Eq. (44), α defined as Eq. (47).

2.3. Example 4. Partial differential equation

The new technique can be readily extended to partial differential equations. Considering the following example

$$\nabla^2 u + \left(\frac{\partial u}{\partial y}\right)^2 = 2y + x^4, \tag{50}$$

$$u(0, y) = 0, \quad u(1, y) = y + a,$$

$$u(x, 0) = ax, u(x, 1) = x(x + a). \tag{50}$$

Its correction variational functional in x and y-directions can be expressed, respectively, as follows:

$$\begin{aligned} u_{n+1}(x, y) = u_n(x, y) &+ \int_0^x \lambda_1 \left[\frac{\partial^2 u_n(\xi, y)}{\partial \xi^2} + \frac{\partial^2 \tilde{u}_n(\xi, y)}{\partial y^2}\right. \\ &\left. + \left(\frac{\partial \tilde{u}_n(\xi, y)}{\partial y}\right)^2 - 2y - \xi^4\right] d\xi \end{aligned} \tag{51}$$

$$\begin{aligned} u_{n+1}(x, y) = u_n(x, y) &+ \int_0^y \lambda_2 \left[\frac{\partial^2 \tilde{u}_n(x, \varsigma)}{\partial x^2} + \frac{\partial^2 u_n(x, \varsigma)}{\partial \varsigma^2}\right. \\ &\left. + \left(\frac{\partial \tilde{u}_n(x, \varsigma)}{\partial \varsigma}\right)^2 - 2\varsigma - x^4\right] d\varsigma, \end{aligned} \tag{52}$$

where $\tilde{u}_n$ is a restricted variation.

The Lagrange multipliers can be easily identified:

$$\lambda_1 = \xi - x, \quad \lambda_2 = \varsigma - y. \tag{53}$$

The iteration formulae in x- and y-directions can be, therefore, expressed, respectively, as follows:

$$\begin{aligned} u_{n+1}(x, y) = u_n(x, y) &+ \int_0^x (\xi - x) \left[\frac{\partial^2 u_n(\xi, y)}{\partial \xi^2} + \frac{\partial^2 u_n(\xi, y)}{\partial y^2}\right. \\ &\left. + \left(\frac{\partial u_n(\xi, y)}{\partial y}\right)^2 - 2y - \xi^4\right] d\xi \end{aligned} \tag{54}$$

$$\begin{aligned} u_{n+1}(x, y) = u_n(x, y) &+ \int_0^y (\varsigma - y) \left[\frac{\partial^2 u_n(x, \varsigma)}{\partial x^2} + \frac{\partial^2 u_n(x, \varsigma)}{\partial \varsigma^2}\right. \\ &\left. + \left(\frac{\partial u_n(x, \varsigma)}{\partial \varsigma}\right)^2 - 2\varsigma - x^4\right] d\varsigma. \end{aligned} \tag{55}$$

To ensure the approximations satisfy the boundary conditions at $x = 1$ and $y = 1$ we modify the variational iteration formulae in x- and y-directions as follows:

$$\begin{aligned} u_{n+1}(x, y) = u_n(x, y) &+ \int_1^x (\xi - x) \left[\frac{\partial^2 u_n(\xi, y)}{\partial \xi^2} + \frac{\partial^2 u_n(\xi, y)}{\partial y^2}\right. \\ &\left. + \left(\frac{\partial u_n(\xi, y)}{\partial y}\right)^2 - 2y - \xi^4\right] d\xi \end{aligned} \tag{56}$$

$$\begin{aligned} u_{n+1}(x, y) = u_n(x, y) &+ \int_1^y (\varsigma - y) \left[\frac{\partial^2 u_n(x, \varsigma)}{\partial x^2} + \frac{\partial^2 u_n(x, \varsigma)}{\partial \varsigma^2}\right. \\ &\left. + \left(\frac{\partial u_n(x, \varsigma)}{\partial \varsigma}\right)^2 - 2\varsigma - x^4\right] d\varsigma. \end{aligned} \tag{57}$$

Now we begin with an arbitrary initial approximation: $u_0 = A + Bx$, where A and B are constants to be determined, by the variational iteration formula in x-direction (54), we have

$$u_1(x, y) = A + Bx + \int_0^x (\xi - x)[0 + 0 - 2y - \xi^4]\, d\xi = A + Bx + x^2 y + \frac{1}{30}x^6. \quad (58)$$

By imposing the boundary conditions at $x = 0$ and $x = 1$ yields $A = 0$ and $B = a - \frac{1}{30}$, thus we have

$$u_1(x, y) = x(xy + a) + \tfrac{1}{30}x(x^5 - 1). \quad (59)$$

By Eq. (56), we have

$$u_2(x, y) = x(xy + a) + \frac{1}{30}x(x^5 - 1) + \int_1^x (\xi - x) \times [2y + \xi^4 + 0 + \xi^4 - 2y - \xi^4]\, d\xi = x(xy + a), \quad (60)$$

which is an exact solution.

The approximation can also be obtained by y-direction or by alternate use of x- and y-directions iteration formulae.

3. Comparison with Adomian's decomposition method [12, 13]

In the Adomian's method [12, 13], the linear term (L) in Eq. (1) is always decomposed into $H + R$, where H is the highest-ordered differential, R is the remainder of the linear operator, as a result, Eq. (1) can be expressed in Adomian's form

$$Hu + Ru + Nu = g. \quad (61)$$

Because H is easily invertible, solving Eq. (61) yields

$$u = \sum_{i=0}^{\infty} \bar{u}_i = u_0 - H^{-1}Ru - H^{-1}Nu + H^{-1}g, \quad (62)$$

with definition

$$\bar{u}_1 = -H^{-1}Ru_0 - H^{-1}A_0,$$

$$\bar{u}_2 = -H^{-1}R\bar{u}_1 - H^{-1}A_1,$$

$$\bar{u}_{n+1} = -H^{-1}R\bar{u}_n - H^{-1}A_n,$$

in which H^{-1} is an inverse operator of H, A_n are called Adomian polynomials, defined as

$$A_0 = N(u_0),$$

$$A_1 = \bar{u}_1 \frac{d}{du_0} N(u_0),$$

$$A_2 = \bar{u}_2 \frac{d}{du_0} N(u_0) + \frac{\bar{u}_1^2}{2!}\frac{d^2}{du_0^2} N(u_0).$$

To compare with the variational iteration method, we construct a correction functional as follows:

$$u_{n+1} = u_n + H^{-1}\lambda\{Hu_n + R\tilde{u}_n + N(\tilde{u}_n)\}, \quad (63)$$

where $\tilde{u}_n$ is considered as a restricted variation.

If H is the first-order derivative, then the multiplier can be easily determined as $\lambda = -1$, resulting in

$$u_{n+1} = u_n - H^{-1}\{Hu_n + Ru_n + Nu_n\} \quad (64)$$

which corresponds to Adomian's formula (62), so the results are the same as Adomian's.

To demonstrate clearly, we begin with $u_0(x) = u(0)$, by Eq. (64), we have

$$u_1 = u_0 - H^{-1}\{Hu_0 + Ru_0 + N(u_0)\} = u_0 + \tilde{u}_1,$$

where

$$u_0 = u(0, x), \quad Hu_0 = Hu(0, x) = 0,$$

$$\bar{u}_1 = -H^{-1}(Hu_0 + Ru_0) - H^{-1}(Nu_0) = -H^{-1}Ru_0 - H^{-1}A_0.$$

By the same manipulation, we have the following second-order approximation

$$u_2 = u_0 + \bar{u}_1 - H^{-1}\{L(u_0 + \bar{u}_1) + R(u_0 + \bar{u}_1) + f(u_0 + \bar{u}_1)\} = u_0 + \bar{u}_1 + \bar{u}_2,$$

where

$$\bar{u}_2 = -H^{-1}\{H(u_0+\bar{u}_1)+R(u_0+\bar{u}_1) + N(u_0+\bar{u}_1)\}.$$

The non-linear term N can be expressed by Taylor series

$$N(u_0+\bar{u}_1) = N(u_0) + \bar{u}_1 \frac{d}{du_0} N(u_0) + \cdots .$$

Therefore we have

$$\begin{aligned}\bar{u}_2 &= -H^{-1}R\bar{u}_1 - H^{-1}A_1 - H^{-1} \times \{H\bar{u}_1 + Ru_0 + N(u_0)\}\\ &= -H^{-1}R\bar{u}_1 - H^{-1}A_1 - H^{-1} \times \{H(-H^{-1}Ru_0 - H^{-1}A_0) + Ru_0 + N(u_0)\}\\ &= -H^{-1}R\bar{u}_1 - H^{-1}A_1 - H^{-1} \times \{-Ru_0 - A_0 + Ru_0 + N(u_0)\}\\ &= -H^{-1}R\bar{u}_1 - H^{-1}A_1\end{aligned}$$

and the nth approximation can be obtained as follows:

$$u_n = \sum_{i=0}^{n} \bar{u}_i,$$

which is the same as Adomian's

So it can be seen clearly that when we begin with initial conditions as initial approximation, and apply restricted variations to all variables except the highest-ordered differential in the correction functional, the above iteration process corresponds to those of Adomian's. However, in the variational iteration method, the multiplier, or the weighted function, can be optimally determined via variational theory [1] instead of simply setting $\lambda = -1$, leading to poor approximation. Furthermore, the restricted variations are applied only to non-linear terms; as pointed out above, the lesser the application of restricted variations the faster the approximations converging to its exact solution.

4. Conclusions

In this paper we have studied few problems with the variational iteration method, which does not require small parameter in an equation as the perturbation techniques do. The results show that

(1) A correction functional can be easily constructed by a general Lagrange multiplier, and the multiplier can be optimally identified by variational theory. The application of restricted variations in correction functional makes it much easier to determine the multiplier.

(2) The initial approximation can be freely selected with unknown constants, which can be determined via various methods.

(3) The approximations obtained by this method are valid not only for small parameter, but also for very large parameter, furthermore their first-order approximations are of extreme accuracy.

(4) Comparison with Adomian's method reveals that the approximations obtained by the proposed method converge to its exact solution faster than those of Adomian's.

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