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Simple genetic algorithm search for critical non-circular failure surface in slope stability analysis

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Abstract

Soil slope stability problems in engineering works are usually analysed using limit equilibrium methods. A number of existing methods are based on finding the critical circular failure surface for homogeneous soils, but failure surfaces tend to be non-circular for layered slopes. A simple genetic algorithm is presented to search the critical non-circular failure surface in slope stability analysis and is used to solve the Morgenstern–Price method to find the factor of safety. The pseudo-static horizontal and vertical forces due to earthquake and surcharge load due to existing buildings and structures on natural slopes are included in the analysis.

1. Introduction

There are many different ways to compute the factor of safety of earth dams or natural slopes including limit equilibrium, finite element and finite difference methods. In recent years the finite element method has been used for slope stability analysis, but limit equilibrium methods are still common practice.

Many methods have been presented to compute the factor of safety using limit equilibrium with a circular failure surface [3,9,12]. A simple circular failure surface method is sufficient for a slope in a homogenous soil layer, while for a heterogeneous multi-soil layers slope, a non-circular failure surface method should be considered as circular methods can over predict the factor of safety. Limit equilibrium has also been used for non-circular failure surfaces [1,2,4,5,8,11], and some of these methods are summarized below.

Nguyen [11] developed a method where the factor of safety is formulated as a multivariate function $F(x)$ with the independent variables x describing the geometry of the failure surface. He employed the simplex method as the optimisation technique. Celestina and Duncan [5] used the same approach for non-circular failure surfaces, but used the alternating-variable optimisation technique. Li and White [8] proposed a more efficient one-dimensional optimisation technique to replace the quadratic interpolation method, which Celestina and Duncan [5] used in the alternating-variable technique. Baker [1] defined the failure surface by a number of nodal points connected by straight lines. The vertical coordinates of the nodal points are the variables in Baker's method and the dynamic programming technique is employed as the optimisation method. Bolton et al. [4] defined a global optimisation algorithm for finding the critical failure surface by nodal points connected by straight lines for any shape of failure. Bardet and Kapuskar [2] presented a simple method of optimisation to search the critical failure surface using the downhill simplex algorithm. A large number of computations are needed to find the critical failure surface, as an arbitrary nodal coordinate could be irrelevant among the rest of created nodal coordinates. For example, a fluctuated failure surface shape could be created using nodal coordinates leading to an unrealistic failure surface (Fig. 1).

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Nomenclature

α_i failure-line slope (slope of base of slice)
α_{f_i} failure-line slope of slice with respect to vertical
$\Delta\alpha_{f_i}$ difference between two subsequent base slice failure slopes ($\Delta\alpha_{f_2} = \alpha_{f_2} - \alpha_{f_1}$)
α_{h_i} failure-line slope of slice with respect to horizontal
a_{h} pseudo-static horizontal coefficient of earthquake
a_{v} pseudo-static vertical coefficient of earthquake
β inclination of the chord connecting the two ends of the natural slope
b width of slice
c' cohesion (for effective stress)
F factor of safety
ϕ' angle of shearing resistance (for effective stress)
ϕ'_{m} mobilized angle of shearing resistance ($\tan\phi'_{\mathrm{m}} = \tan\phi'/F$)
h_Q height of intersection of resultant inter-slice forces
h_G height of centre of gravity of slice
N_{slice} total number of slices, default value is 150
γ bulk density
$\bar{\gamma}_i$ average bulk density of each slice
θ_i slope of resultant (Q_i) of pair of inter-slice forces
η_i slope of inter-slice force Z_i
η_{i+1} slope of inter-slice force Z_{i+1}
P total reaction normal to base of slice
P' reaction (due to effective stress) normal to base of slice
Q_i resultant of pair of inter-slice forces Z_i and Z_{i+1}
q surcharge along the slope
S total shear force available
u pore-pressure
S_{m} total shear force mobilized
W weight of slice
x_{q_i}, y_{q_i} coordinates of acting point of inter-slice resultant forces, Q_i
x'_i normalized coordinate of slice i
X_{e_i}, Y_{e_i} coordinates of mid point of slice base
X_{g_i}, Y_{g_i} coordinates centre of gravity of slice
X_{s_i}, Y_{s_i} coordinates of left side of slice at ground surface
$X_{s_{(i+1)}}, Y_{s_{(i+1)}}$ coordinates of right side of slice at ground surface
X_{f_i}, Y_{f_i} coordinates of left side of slice on failure surface
$X_{f_{(i+1)}}, Y_{f_{(i+1)}}$ coordinates of right side of slice on failure surface
Z_i inter-slice force on left side of slice i
Z_{i+1} inter-slice force on right side of slice i

In this study, instead of searching and optimising along nodal y-coordinates, the search through failure-line slopes using a simple genetic algorithm is presented. The search through failure-line slopes is much more efficient and quicker to solve than nodal y-coordinates searching, because the slope of the failure surface of each slice (α in Fig. 2) is related to the slope of adjacent slices. In a slope stability analysis where the failing mass moves from left to right, the angle of the base of a slice (α) is usually shifted counter-clockwise when moving

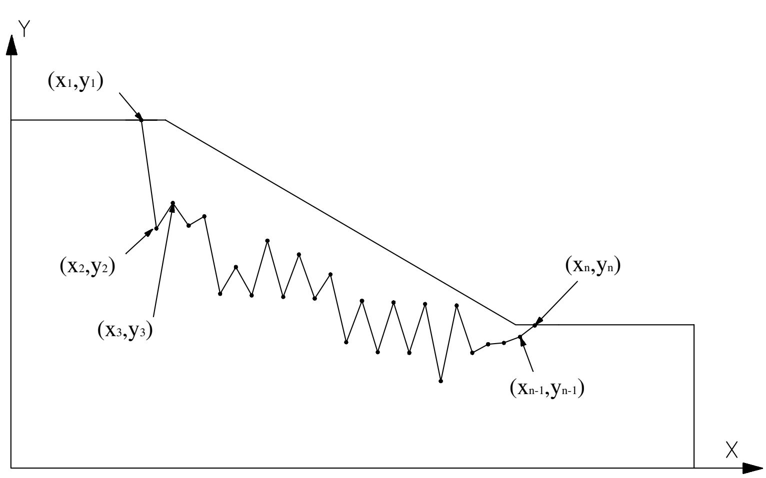

Fig. 1. A fluctuating failure surface as a case in a Nodal Optimization Method.

from left to right (e.g., Fig. 2). Searching for a failure surface using nodal y-coordinates cannot include this aspect easily, possibly resulting in an unrealistic failure surface as shown in Fig. 1.

This paper presents a simple computation format of the Morgenstern–Price method for the non-circular slope stability analysis with pseudo-static horizontal and vertical forces due to earthquake loading. The main reason for using the Morgenstern–Price is that it is a so-called rigorous solution (i.e., rigorous in that both force and moment equilibrium are satisfied if one can make certain assumptions), and produces realistic answers for surfaces which require significant internal distortion of the sliding mass of soil. The option of surcharge loading is also included, which can be used to model the effect of buildings on slope stability. An important feature is that no assumptions are required with regards to the geometry of the failure surface and no restrictions are placed on the positions of the initiation and termination point of the failure surface.

The simple genetic algorithm used in this study has two purposes: firstly to find the critical non-circular failure surface in finite or infinite slopes, and secondly to solve the Morgenstern–Price method to find the factor of safety corresponding to the critical failure surface. As circular failure surfaces are a subset of more general non-circular surfaces, the proposed method will find a circular failure surface if this is the critical failure surface for the particular problem.

2. Presenting a simple solution for Morgenstern–Price method

Although similar to the Spencer method [12], the Morgenstern–Price method [10] was selected for the analysis using the simple computation format.

Fig. 2 shows a natural slope with a head to toe angle of β. Fig. 3 shows details of inter-slice forces for the slice number i. As described in more detail in Appendix A, the resultant of interslice forces in each slice can be written as follows:

$$Q_i = \frac{\frac{c'\cdot b\cdot\sec\alpha}{F} + \frac{\tan\phi'}{F}(W\cdot\cos\alpha - W\cdot a_{\mathrm{v}}\cdot\cos\alpha - W\cdot a_{\mathrm{h}}\cdot\sin\alpha - u\cdot b\cdot\sec\alpha + q\cdot b\cdot\cos\alpha) - W\cdot\sin\alpha + W\cdot a_{\mathrm{v}}\cdot\sin\alpha - W\cdot a_{\mathrm{h}}\cdot\cos\alpha - q\cdot b\cdot\sin\alpha}{\cos(\alpha-\theta_i)\cdot(1+\tan(\alpha-\theta_i)\cdot\frac{\tan\phi'}{F})} \quad (1)$$

In order to satisfy equilibrium equations, the summation of resultant interslice forces and overall moment over an optional point must be zero. In this case the moments about the origin ($x = 0$, $y = 0$) are taken to be zero:

$$\sum(Q_i\cdot\cos\theta_i) = 0 \quad (2)$$

$$\sum(Q_i\cdot\sin\theta_i) = 0 \quad (3)$$

$$\sum(M) = \sum(Q_i\cdot\cos\theta_i\cdot Y_{q_i} + Q_i\cdot\sin\theta_i\cdot X_{q_i}) = 0 \quad (4)$$

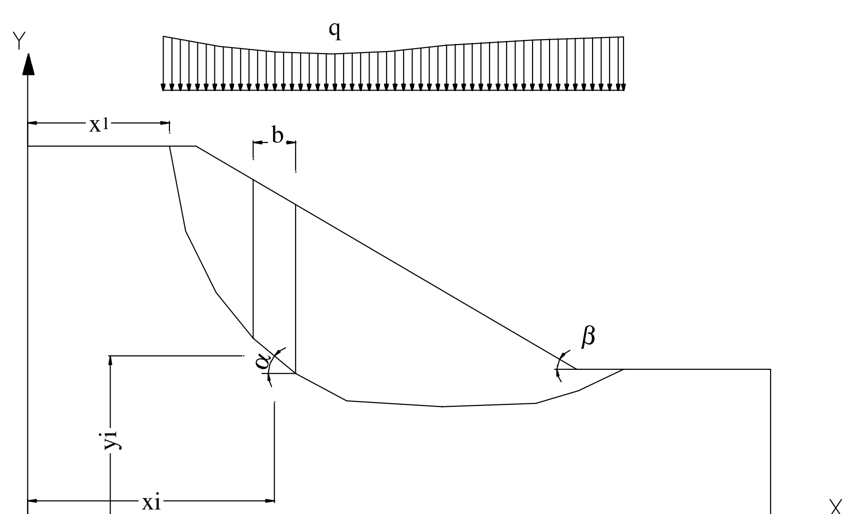

Fig. 2. General failure surface in a slope stability analysis.

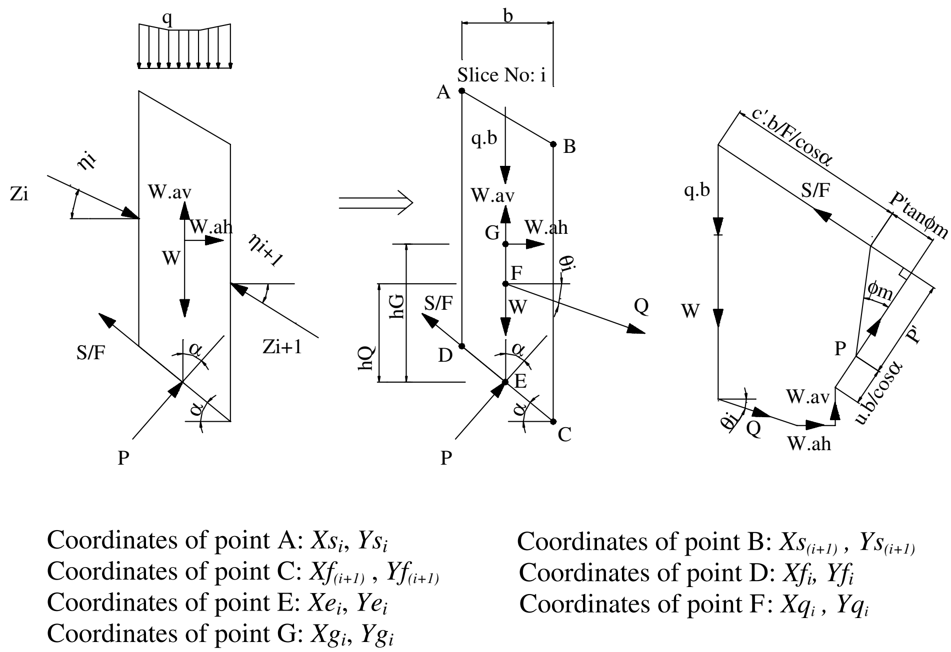

Fig. 3. (a)–(c) Inter-slice forces in slice number i. Coordinates of point A: X_{s_i}, Y_{s_i}. Coordinates of point B: $X_{s_{(i+1)}}, Y_{s_{(i+1)}}$. Coordinates of point C: $X_{f_{(i+1)}}, Y_{f_{(i+1)}}$. Coordinates of point D: X_{f_i}, Y_{f_i}. Coordinates of point E: X_{e_i}, Y_{e_i}. Coordinates of point F: X_{q_i}, Y_{q_i}. Coordinates of point G: X_{g_i}, Y_{g_i}.

To find the factor of safety, Eq. (2) or (Eq. (3)) and Eq. (4) need to be solved. There will be two equations and two unknowns such as λ and F.

If it is assumed a pair of (F^*, λ^*) is one of the answers to the two equations $f(F, \lambda) = 0$ and $g(F, \lambda) = 0$, then (F^*, λ^*) will also be an answer to the following equation:

$$|f(F,\lambda)| + |g(F,\lambda)| = 0 \tag{5}$$

Using the above algebraic theory, Eqs. (2) and (4) can be written as follows:

$$\begin{aligned}\left|\sum M\right| + \left|\sum(Q_i \cdot \cos\theta_i)\right| &= \left|\sum(Q_i \cdot \cos\theta_i \cdot Y_i + Q_i \cdot \sin\theta_i \cdot X_i)\right| \\ &\quad + \left|\sum(Q_i \cdot \cos\theta_i)\right| = 0\end{aligned} \tag{6}$$

Eq. (6) is now solved using the simple genetic algorithm that is explained in Section 3.1.

3. Simple genetic algorithm

The simple genetic algorithm refers to a model introduced and investigated by Holland [7] which uses the concepts of genetics in a specific way as an optimisation tool. A simple genetic algorithm (SGA) uses strings of binary coding, 0 and 1, to encode whatever information is needed to define a distinct solution to a problem. This solution may then be tested to produce a fitness value. For example, if the goal is to find three unknown values such as x, y, z, then each chromosome will be a string of binary digits of x, y, and z. Comparison may be made with biological coding, which uses units with four possible values in DNA. Clearly, real variables such as coordinates need to be expressed as integer values by breaking up the possible range into a number of steps, so for example $x_{\text{int}} = \text{round}(256(x - x_{\min})/(x_{\max} - x_{\min}))$ for an eight bit binary code.

The simple genetic algorithm works in two main stages: creating the initial population and reproduction. First the initial population is created and each number is stored in a chromosome in the binary format. A fitness value associated with each chromosome is calculated. The population is then sorted in descending or ascending order according to the fitness value. A classical way of ensuring that the best solution is never lost is to copy the best individual to the next generation. Then half of the population is selected for the reproduction process. A crossover process with a probability of 0.7–0.9 is applied to two selected parent chromosomes.

(a): chromosome of F and λ

<table>
<tr><td colspan="10">F</td><td colspan="7">λ</td></tr>
<tr><td>1</td><td>1</td><td>0</td><td>1</td><td>0</td><td>1</td><td>1</td><td>0</td><td>1</td><td>1</td><td>1</td><td>0</td><td>1</td><td>0</td><td>0</td><td>1</td><td>1</td></tr>
</table>

(b): chromosome of x 1, αf1, Δαf2, Δαf3,… , Δαfn:

<table>
<tr><td colspan="11">X1</td><td colspan="6">αf1</td><td colspan="4">Δαf2</td><td colspan="4">Δαf3</td><td>…</td><td colspan="4">Δαfn</td></tr>
<tr><td>1</td><td>1</td><td>0</td><td>1</td><td>0</td><td>0</td><td>1</td><td>0</td><td>1</td><td>1</td><td>0</td><td>0</td><td>1</td><td>1</td><td>0</td><td>1</td><td>0</td><td>1</td><td>1</td><td>0</td><td>0</td><td>0</td><td>1</td><td>1</td><td>1</td><td>…</td><td>1</td><td>0</td><td>1</td><td>0</td></tr>
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Fig. 4. Examples of two chromosomes.

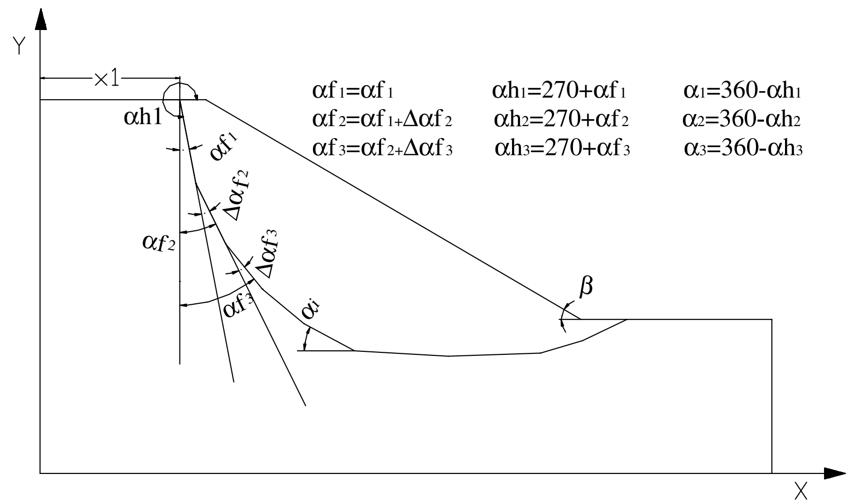

Fig. 5. Non-circular failure surface.

A random position along the length of the chromosome is selected and the values of each binary string are exchanged or crossed by swapping all characters after this position. The two new chromosomes created are known as children of those parents. Mutation is applied to a small proportion of chromosomes, thus introducing the possibility of significant shifts away from the solutions currently being converged on, that overcomes problems associated with local maxima or minima in analyses. Each binary value in a chromosome selected for mutation is swapped with a probability of 0.008, i.e., each 0 or 1 is changed into 1 or 0 with a probability

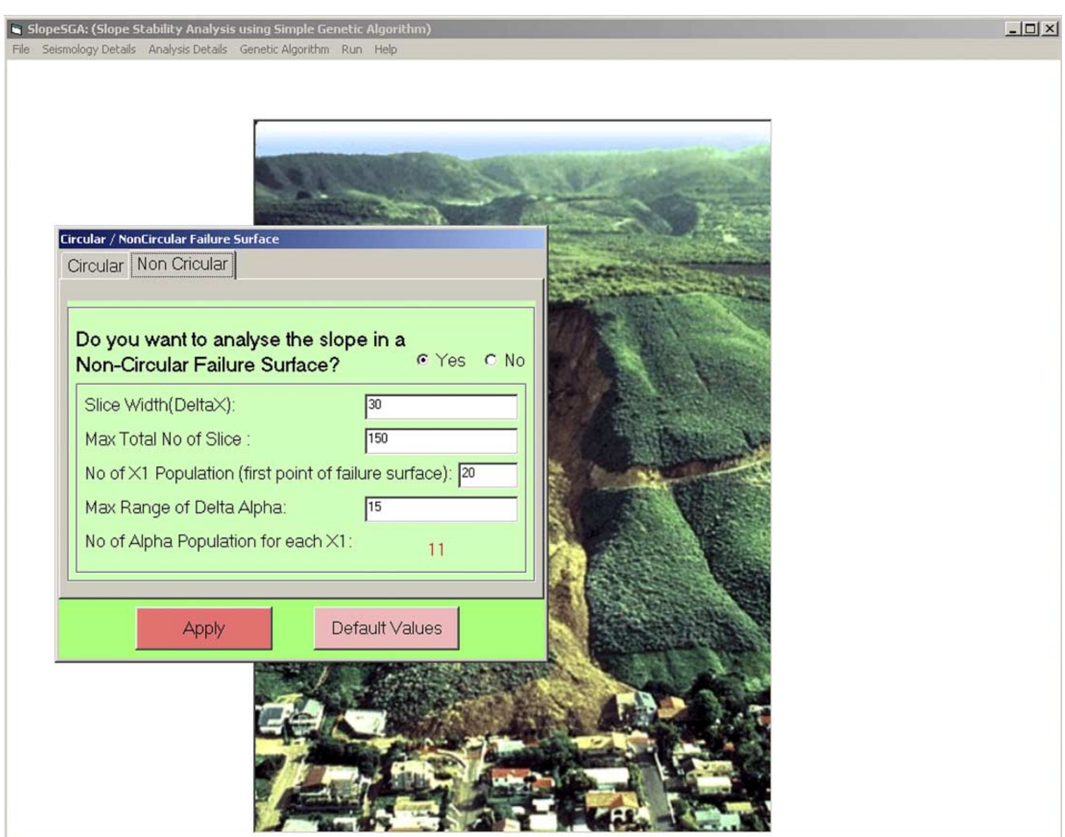

Fig. 6. Slope stability analysis program (SlopeSGA) in VB6.

of 8/1000. The fitness values of the new population, which include both children and parent chromosomes, are then calculated.

The process of reproduction, crossover, mutation and evaluation is repeated as a cycle of generation. A number of cycles are performed until an optimal solution is determined. Goldberg [6] refers to this basic implementation as a simple genetic algorithm (SGA). The selection of the part of each generation to go into the reproduction process means that the population as a whole tends to get fitter. This might mean that the population tends towards being identical, or if there are a number of solutions with a similar fitness, then the population can converge on all the solutions at the same time. It may be necessary to adjust the parameters of the search to ensure that local maxima (or minima, in this case) are not rejected too early.

McCombie and Wilkinson [9] developed a simple genetic algorithm to search for the minimum factor of safety of a circular failure surface in slope stability analysis. They presented a three-dimensional chromosome coding containing the x and y coordinates of the centre of a circle and the radius of a circular failure surface. They also showed that replacing the radius with a tangent level, or with the coordinates of a point the circle had to pass through (thus creating a four

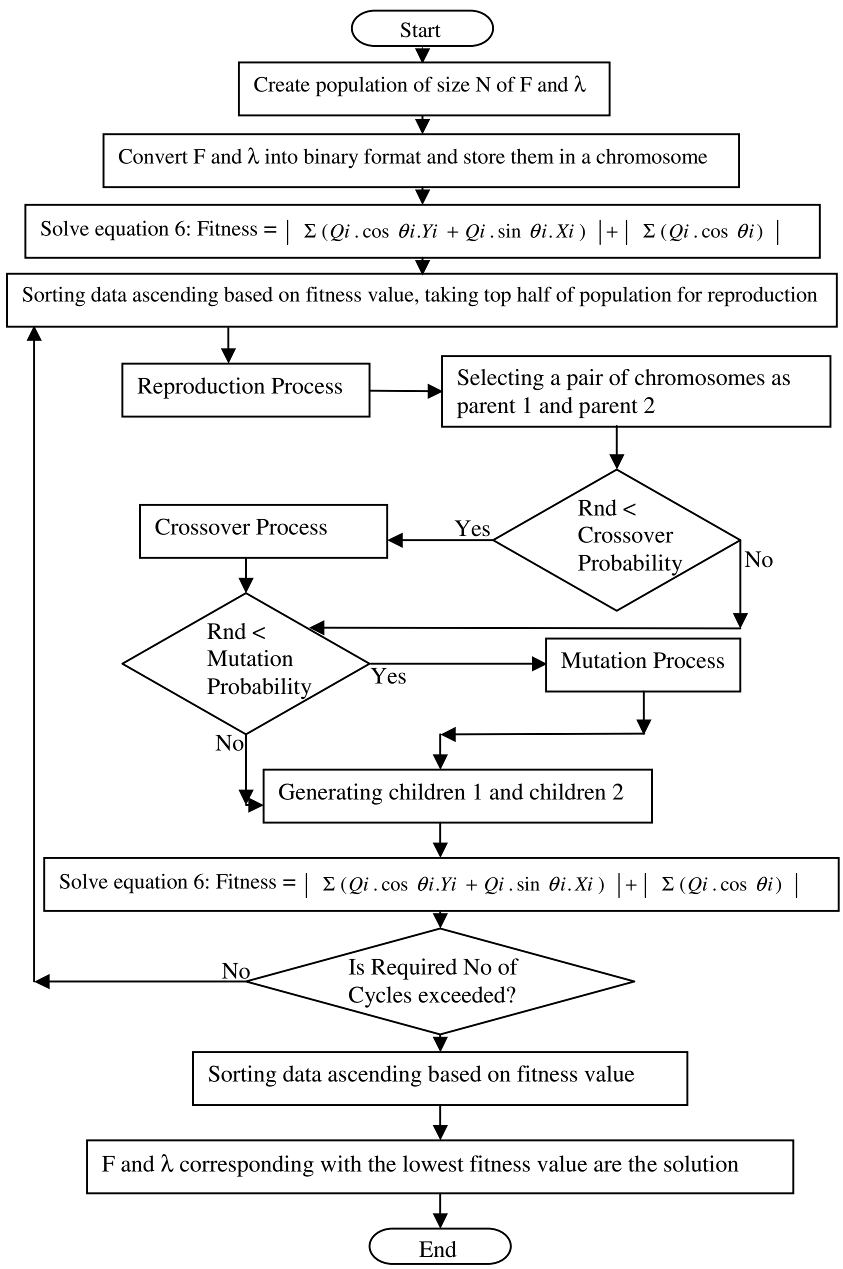

Fig. 7. Simple genetic algorithm to find factor of safety.

dimensional search space), would usually work better, as the formulation of the problem becomes closer to what determines the fitness of each chromosome. In addition, the use of a simple genetic algorithm was found to be more efficient at solving slope stability problems than "traditional" numerical optimisation methods. While this previous research is only applicable to circular failure surfaces, the solution algorithm can be extended to non-circular failure surfaces. In this case definition of the surface by the change of angle at points along it is a more natural definition of the problem than using y co-ordinates, and can, therefore, be expected to produce more efficient convergence by preventing unrealistic failure surfaces similar to those shown in Fig. 1.

3.1. Using the simple genetic algorithm to solve Eq. (6) in order to find the lowest factor of safety of a non-circular failure surface

Eq. (6) shows there are two unknown variables (θ_i and F). Using Eq. (14) or (15), θ_i can be replaced with λ. As in any numerical solution technique, the initial population (values) of F and λ has a considerable effect

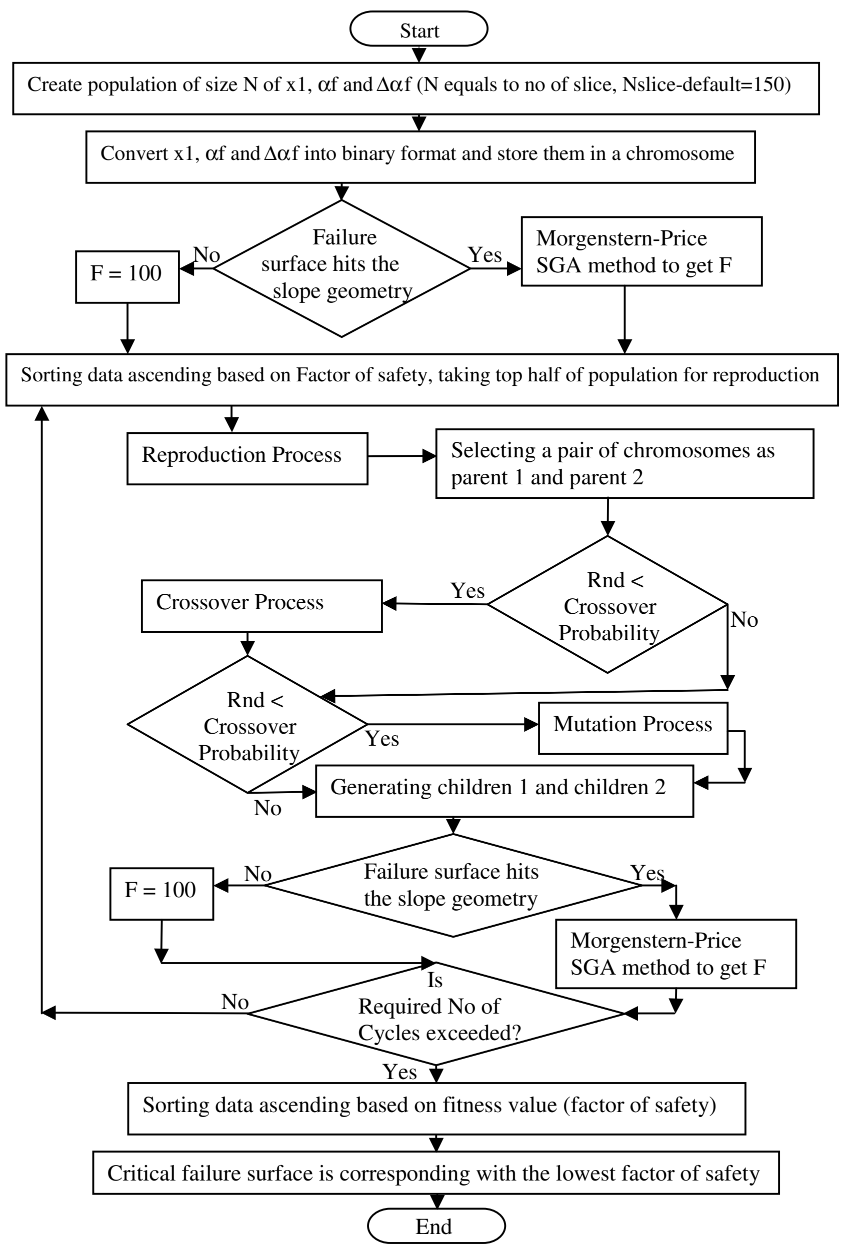

Fig. 8. Simple genetic algorithm to find non-circular failure surface.

on the convergence rate. The initial value of λ is assumed to be approximately $0.7\tan\beta$ [13].

Let N be the size of the population for F and λ, and 0.5 the percentage of reproduction. In order to easily convert F into binary code, in the initial population, values of F are created in a range of 1–1000 as integer values. Values of λ are created in a range around $100{*}0.7\tan\beta$ ($0.7\tan\beta$ is a real value). These values are converted into binary format and stored in a string as a chromosome (Fig. 4). The values of F and λ are divided by 100 to obtain their real values in the calculation. The fitness value is calculated for every combination of F and λ in the population using Eq. (6).

$$\text{Fitness} = \left|\sum(Q_i \cdot \cos\theta_i \cdot Y_i + Q_i \cdot \sin\theta_i \cdot X_i)\right| + \left|\sum(Q_i \cdot \cos\theta_i)\right|$$

The chromosomes are sorted into ascending order according to their fitness value and the half with the lowest fitness value is selected for the reproduction process. The crossover and mutation are applied to chromosomes during the reproduction process as described above. The population is subjected to a number of genetic cycles in order to find the F and λ that minimize the fitness value. All these calculations are performed in a slope stability analysis program, SlopeSGA that is

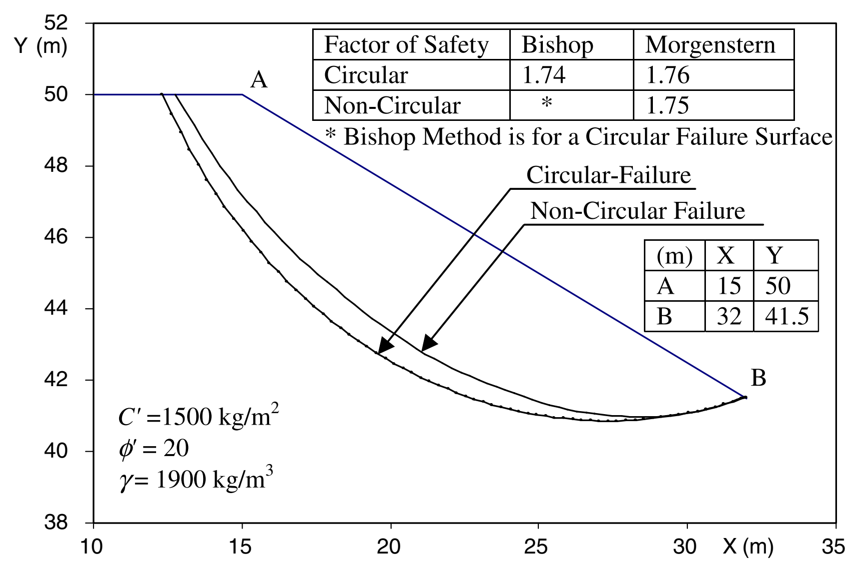

Fig. 9. An example of a natural slope with a homogenous soil layer.

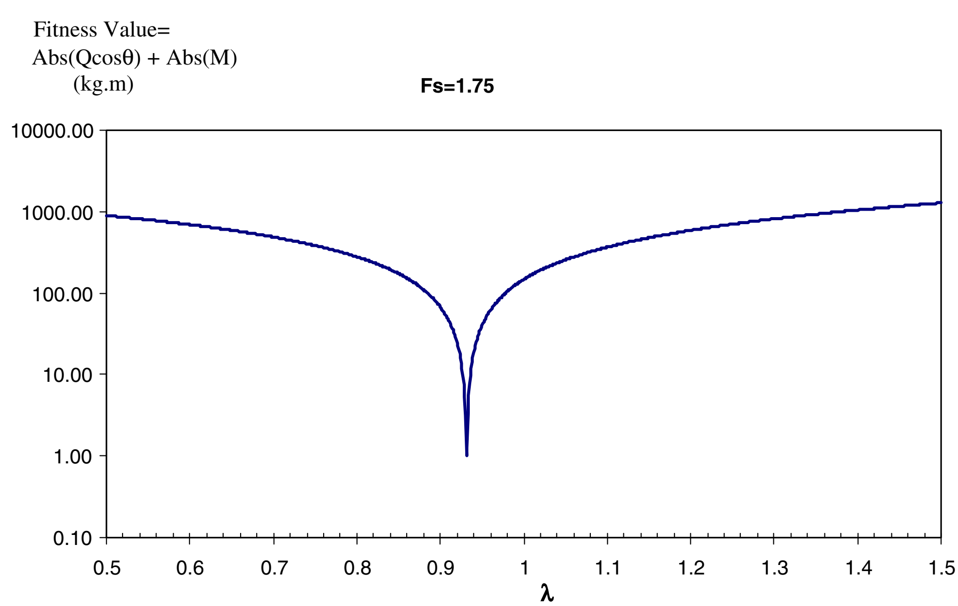

Fig. 10. Converging the fitness value in Morgenstern–Price method.

written in Visual Basic 6 by the first author, as depicted in Figs. 6 and 7.

3.2. Using the simple genetic algorithm to find critical non-circular failure surface

As the initial part of the simple genetic algorithm method, a population of all searching parameters needs to be created. The first x coordinate of the failure surface, x_1, is created randomly. Let Nx_1 be the number of populations for x_1. Therefore, Nx_1 cases of x_1 are created randomly. Now let $N\alpha_f$ be the number of populations for α_f for each x_1. Therefore, $N\alpha_f$ cases of α_{f_1} are randomly created for each x_1. The initial value of α_{f_1} is chosen randomly around the Rankine failure angle range. As the Rankine failure angle with respect to the horizontal is $45 + \phi/2$, the α_{f_1} range will be $45 - \phi/2$ or 30–45, for ϕ equals to 30 and 0, respectively (Fig. 5). These two values of x_1 and α_{f_1} are converted into binary code and stored in a chromosome.

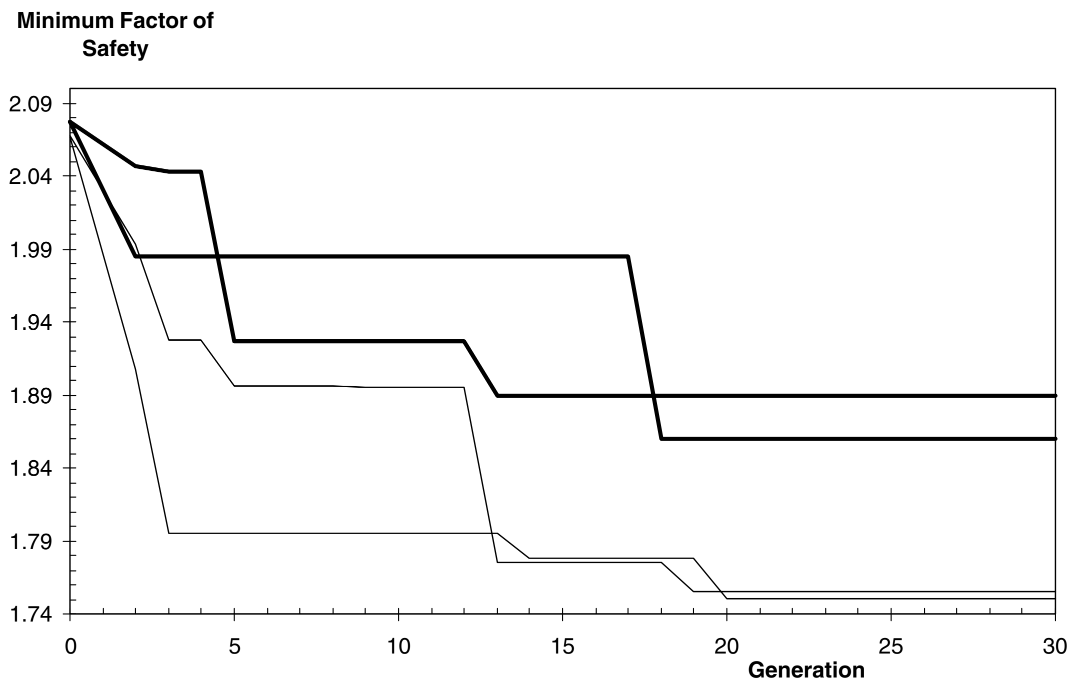

Fig. 11. A comparison between random reproduction and the SGA.

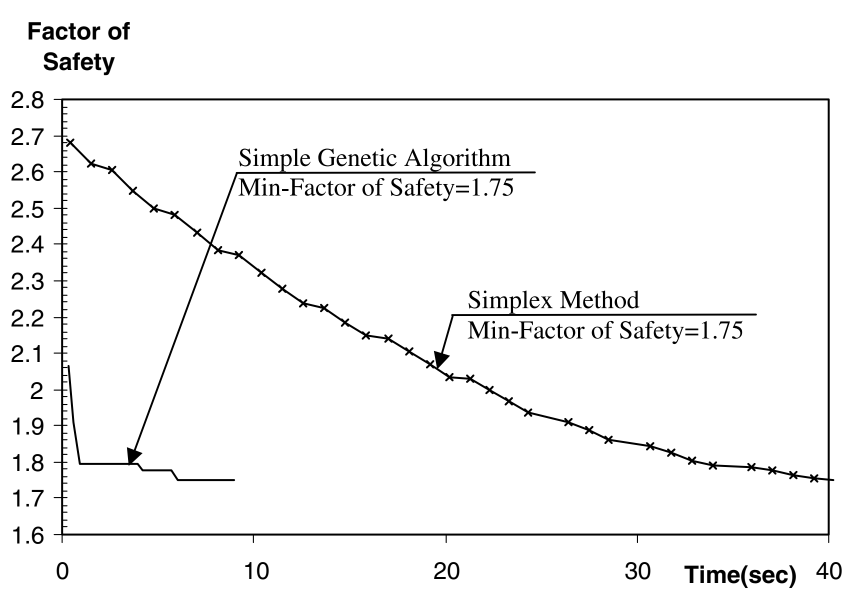

Fig. 12. A comparison between the SGA and simplex method.

After choosing x_1 and α_{f_1} for a failure surface, the rest of the failure-line slopes through the failure surface need to be chosen. As shown in Fig. 5, each failure-line slope has a relation to the previous one, so instead of choosing failure-line coordinates as in previous research, e.g. [1], the angular difference $\Delta\alpha_{f_i}$ between each successive failure-line slope is chosen randomly for all slices. This allows the population of irrelevant failure-lines slopes to be ignored easily.

Now let N_{slice} be the total number of slices for a non-circular failure surface. $\Delta\alpha_{f_i}$ is randomly selected through a reasonable range. In order to start with a sufficient population, 11 different categories of $\Delta\alpha_{f_i}$ range are defined. For example, the failure-lines slopes can be very rapid ($\Delta\alpha_{f_i}$ between 5° and 15°), rapid ($\Delta\alpha_{f_i}$ between 0° and 10°), gentle ($\Delta\alpha_{f_i}$ between 0° and 5°), and very gentle ($\Delta\alpha_{f_i}$ between 0° and 3°). More cases are created, such as the failure surface continues horizontally

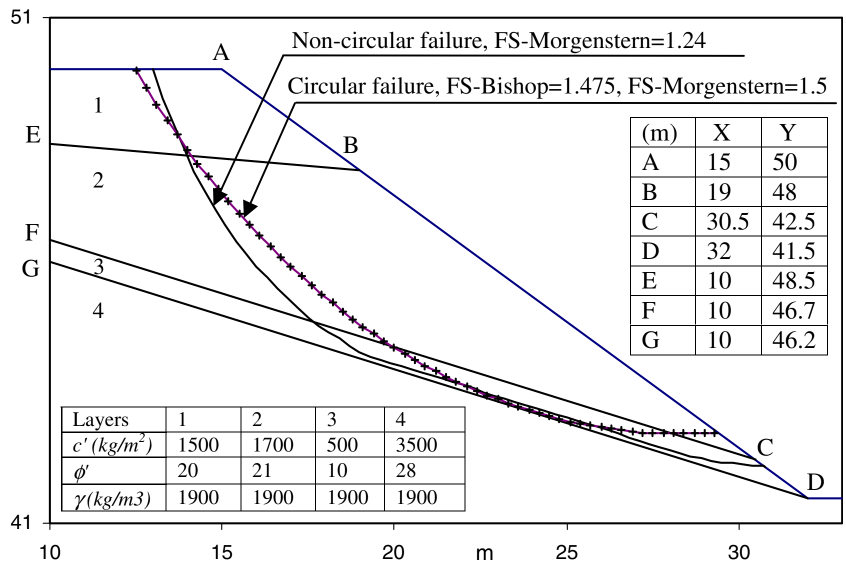

Fig. 13. Comparison of a circular with a non-circular failure surface method.

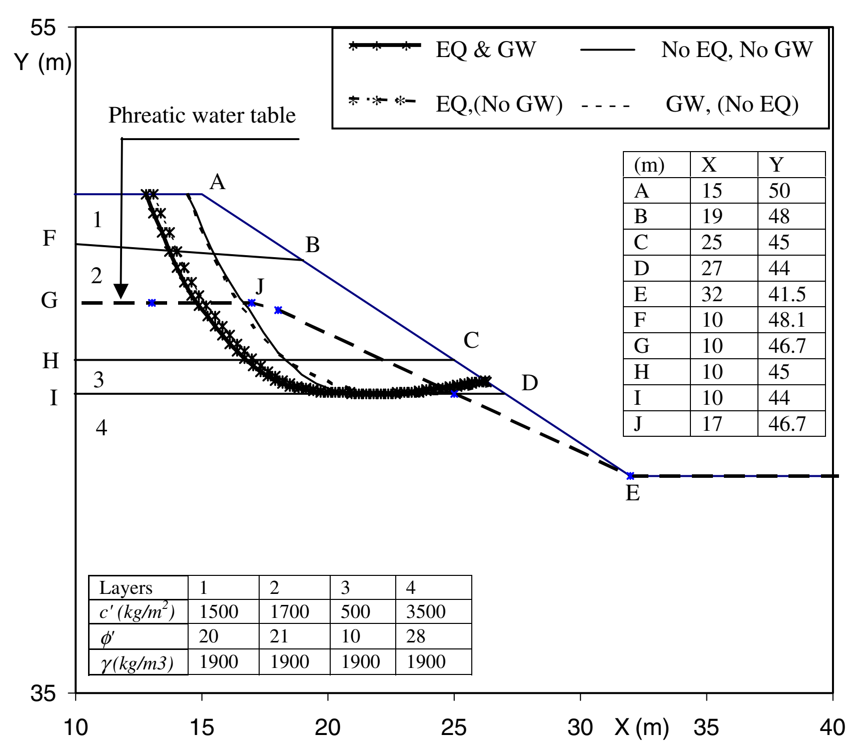

Fig. 14. Example of a natural slope with a complex soil layers.

Table 1
Results of slope stability analysis in example 3

Loading	Method	No. of slice	Factor of safety (Morgenstern–Price)
No earthquake – No ground water	Non-circular	40	1.48
$a_h = 0.1$ – No ground water	Non-circular	44	1.37
No earthquake – Ground water	Non-circular	41	1.36
$a_h = 0.1$ – Ground water	Non-circular	45	0.98

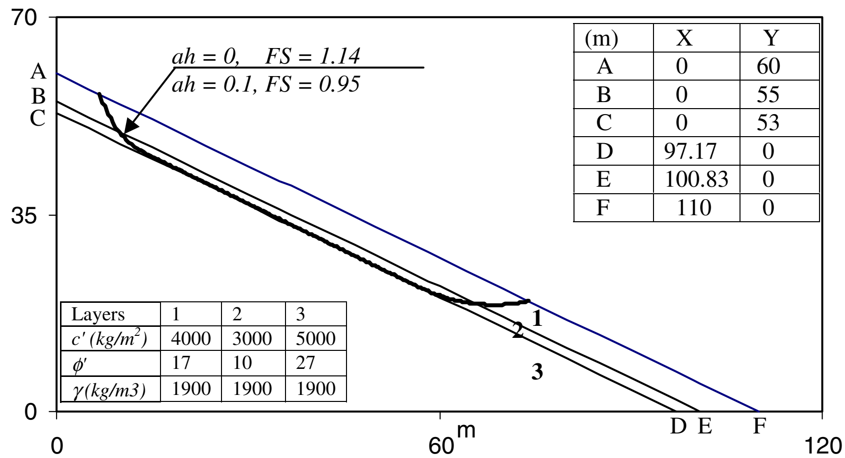

Fig. 15. Results of the infinite slope analysis in example 4.

once the failure-line slope becomes horizontal. The eleven cases make the simple genetic algorithm more successful in finding the critical non-circular failure surface. The values of $\Delta\alpha_{f_i}$ are converted into binary code and stored in the chromosome containing x_1 and α_{f_1} (Fig. 4).

The crossover and mutation are applied to the chromosomes in the reproduction process as described in Section 3. As the chromosome represents a search space of $N_{\text{slice}} + 2$, or usually 152 dimensions (based on the default value where N_{slice} equals to 150), more crossover cases need to be applied in order to generate more new chromosomes. Five different cases of crossover are applied: one point crossover in the x_1 coding of the chromosome, two point crossover in the x_1 and α_{f_1} coding parts of chromosome, one point crossover through the whole chromosome, one point and two points crossover in the $\Delta\alpha_{f_i}$ coding of chromosome.

The fitness value is the factor of safety against failure for the surface, which is calculated using the simple genetic algorithm method, as explained in Section 3.1. If the created non-circular failure surface does not hit the geometry of the natural slope, a large value is given to the fitness value and the failure surface is subsequently excluded from the population through the "survival of the fittest" characteristic of the genetic algorithm. A number of genetic cycles are calculated in order to find the critical non-circular failure surface with the lowest factor of safety.

All these calculations are performed in the slope stability analysis program as depicted in Figs. 6 and 8.

4. Examples

For the purpose of illustration, four examples of natural slopes are analysed using the simple genetic algorithm method proposed in this paper. The aforementioned slope stability analysis program is used to analyse these examples as follows:

(a) An example of a natural slope with a homogenous soil layer, as shown in Fig. 9, is analysed. The factor of safety is calculated for the slope using the Bishop and Morgenstern–Price methods for both circular and non-circular methods. Fig. 10 shows how the fitness value, $\text{abs}(Q\cos q) + \text{abs}(M)$ in Eq. (6), is changed and eventually converged to a negligible value as the solution converges to the minimum factor of safety. The simple genetic algorithm method is also compared to a random reproduction approach in Fig. 11 and to the simplex method in Fig. 12.

In Fig. 11, comparison was made between random reproduction and the simple genetic algorithm over 30 cycles. In the random reproduction, a population of 100 was set, and then the best of 50% of the population is kept after each reproduction. The thick lines show the lowest factor of safety achieved after each set of 100 analyses of randomly generated surfaces (eventual minimum 1.86), whilst the thin lines show the corresponding values with genetic reproduction producing each generation (minimum 1.75). The increased convergence rate for the simple genetic algorithm is clear. The code is written in Visual Basic 6 which is slow compared to other packages, but this does allow easy comparison of the relative efficiency of different methods to by the time taken for each run. The simple genetic algorithm run took 9 s for the 30 generations.

Fig. 12 shows the results from solving the same example using the simplex method presented by Bardet and Kapuskar [2]. In this case this method was modified by the fact that slope of the base of any slice is related to the slope of adjacent slices, as described above. This code was also written in Visual Basic 6, and took 41 s to find the non-circular failure surface with the factor of safety of 1.75. The comparison between the simple genetic algorithm and the simplex method, Fig. 12, shows the rapid convergence of the simple genetic algorithm compared to the simplex method. As the SGA is started with a wider population (i.e., 100 population), therefore, the factor of safety in this method is a lower value at the beginning of the search compare to the simplex method (Fig. 12).

(b) An example of a slope with complex soil layering, as shown in Fig. 13, is analysed using the proposed SGA method. In this example the difference between the factor of safety in circular and non-circular failure surface methods is presented. As expected, the non-circular failure surface is drawn towards the weakest layer, resulting in a lower factor of safety than that for the circular surface.

(c) Another natural slope with complex soil layering, as shown in Fig. 14, is analysed using the presented method. Four loading cases are considered: water pressure and earthquake loading; water pressure and no earthquake loading; earthquake loading and no water pressure; and absence of both water pressure and earthquake loading. A pseudo-static horizontal coefficient of earthquake loading of 0.1 is used in this analysis. The results are presented in Table 1. As expected, an increase in water pressure resulted in a decreased factor of safety, and an increase in pseudo-static horizontal earthquake loading decreased the factor of safety.

(d) An infinite slope, as shown in Fig. 15, is analysed using the simple genetic algorithm. In this example, a soft layer is located between two layers with higher strength. A pseudo-static horizontal coefficient of earthquake loading of 0.1 is assumed.

5. Conclusions

Many previous approaches to determining the non-linear failure surface assumed the slope of the base of any slice is independent of the slope of adjacent slices. However, relating the slopes of adjacent slices results in greatly increased computational efficiency. The presented simple genetic algorithm method can be applied to find the non-circular failure surface with the lowest factor of safety very quickly compared to random or simplex method approaches. The Morgenstern–Price method can be easily solved with the simple genetic algorithm in order to obtain the factor of safety for a variety of slope geometries and loading conditions. The results of this study suggest that the presented searching method could be used in order to analyse the stability of earth dams, finite or effectively infinite natural slopes and any other geotechnical problem with layered or unlayered geology. The option of a surcharge load and pseudo-static horizontal and vertical forces due to earthquake loading are included to enable a comprehensive evaluation of slope stability.

For a slope in a homogenous material, the non-linear algorithm approximates a circular failure surface and predicts a similar factor of safety to that for a circular failure surface. For a slope with a layered structure, the circular methods can over predict the factor of safety, which might lead to unconservative estimates of slope stability. In these cases, non-circular slope stability analysis is essential for reliable assessment of stability.

Appendix A. Complete formulation of the solution for the Morgenstern–Price method

(a) Weight of slice number i:

$$W = \left\{ \frac{(Y_{s_{(i+1)}} - Y_{f_{(i+1)}}) + (Y_{s_i} - Y_{f_i})}{2} \right\} \cdot b \cdot \bar{\gamma}_i \tag{A.1}$$

(b) Total reaction of normal in the base of the slice, P, which can be presented as:
  1. Force (P') due to the effective stress.
  2. Force ($u \cdot b \cdot \sec \alpha$) due to the pore pressure (u)

$$P = P' + u \cdot b \cdot \sec \alpha \tag{A.2}$$

(c) Mobilized shear force ($S_m = S/F$)

$$S = c' b \cdot \sec \alpha + P' \cdot \tan \phi' \tag{A.3}$$

$$S_m = c' \cdot b \cdot \sec \alpha / F + P' \cdot \tan \phi' / F \tag{A.4}$$

(d) Pseudo-static horizontal force due to earthquake, $a_h \cdot W$

(e) Pseudo-static vertical force due to earthquake, $a_v \cdot W$.

(f) Surcharge force in the slice due to surcharge load along the natural slope, $q \cdot b$.

(g) Inter-slice forces Z_i and Z_{i+1} with horizontal angle of η_i and η_{i+1}, respectively.

(h) Q_i, Resultant force of Z_i and Z_{i+1} forces. This resultant force, Q_i that acts with a horizontal angle of θ_i (Fig. 3), is calculated from equilibrium condition in two perpendicular directions, using Eqs. (A.5) and (A.6).

For equilibrium in each slice, the sum of inter-slice forces in the P and S/F direction must be zero. The process is shown in Eqs. (A.5) and (A.6), respectively,

$$P + Q_i \cdot \sin(\alpha - \theta_i) - W \cdot \cos\alpha + W \cdot a_v \cdot \cos\alpha + W \cdot a_h \cdot \sin\alpha - q \cdot b \cdot \cos\alpha = 0 \tag{A.5}$$

$$S/F - Q_i \cdot \cos(\alpha - \theta_i) - W \cdot \sin\alpha + W \cdot a_v \cdot \sin\alpha - W \cdot a_h \cdot \cos\alpha - q \cdot b \cdot \sin\alpha = 0 \tag{A.6}$$

Substituting Eqs. (A.3) and (A.4) in Eqs. (A.5) and (A.6):

$$Q_i = \frac{\frac{c' \cdot b \cdot \sec\alpha}{F} + \frac{\tan\phi'}{F}(W \cdot \cos\alpha - W \cdot a_v \cdot \cos\alpha - W \cdot a_h \cdot \sin\alpha - u \cdot b \cdot \sec\alpha + q \cdot b \cdot \cos\alpha) - W \cdot \sin\alpha + W \cdot a_v \cdot \sin\alpha - W \cdot a_h \cdot \cos\alpha - q \cdot b \cdot \sin\alpha}{\cos(\alpha - \theta_i) \cdot (1 + \tan(\alpha - \theta_i) \cdot \frac{\tan\phi'}{F})} \tag{A.7}$$

For moment equilibrium in each slice, take the moment about the point E equal zero (Fig. 3):

$$\sum M_E = Q \cdot \cos\theta \cdot h_Q - W \cdot a_h \cdot h_G = 0 \tag{A.8}$$

After calculating Q for each slice, h_Q is calculated using Eq. (A.8), then the y coordinate of point F in Fig. 3, Y_{q_i} is:

$$Y_{q_i} = Y_{e_i} + h_Q \tag{A.9}$$

Now for overall equilibrium in the natural slope, the sum of horizontal and vertical inter-slice forces must be zero.

$$\sum (Q_i \cdot \cos\theta_i) = 0 \tag{A.10}$$

$$\sum (Q_i \cdot \sin\theta_i) = 0 \tag{A.11}$$

Furthermore, the sum of the overall moments about an arbitrary point must be zero, in this case let the moments about the origin ($x = 0$, $y = 0$) be zero:

$$\sum (M) = \sum (Q_i \cdot \cos\theta_i \cdot Y_{q_i} + Q_i \cdot \sin\theta_i \cdot X_{q_i}) = 0 \tag{A.12}$$

The vertical inter-slice force divided by the horizontal inter-slice force can be defined in terms of $\lambda \cdot f(x_i')$ (Morgenstern–Price method [10]), where x_i' is the linearly normalized x_i coordinate with values at the two ends of the failure surface equal to zero and π, respectively. In this case, $f(x_i')$ is by convention assumed to be equal to $\sin(x_i')$, therefore, the overall shape of $f(x_i')$ on the failure surface is a half sin shape. In the Spencer method [10], $f(x_i')$ is equal to 1, therefore, the angle of inter-slice resultant force is equal for all the slices through the failure surface, Eq. (A.15). Now let θ_i be the horizontal angle of inter-slice resultant force Q_i.

$$\tan\theta_i = \lambda \cdot f(x_i') \tag{A.13}$$

Two cases for $f(x_i')$ will be as follows:

(a) $f(x_i') = \sin(x_i')$ then $\theta_i = \arctan(\lambda \cdot \sin(x_i'))$ (Morgenstern–Price method) (A.14)

(b) $f(x_i') = 1$ then $\theta_i = \arctan(\lambda)$ (Spencer method) (A.15)

Now to find the factor of safety, Eq. (A.10) (or Eq. (A.11)) and Eq. (A.12) need to be solved. There will be two equations and two unknowns such as λ and F.

Let assume a pair of (F^*, λ^*) is one of the answers to the two equations $f(F,\lambda) = 0$ and $g(F,\lambda) = 0$, therefore, (F^*, λ^*) will also be an answer to the following equation:

$$|f(F,\lambda)| + |g(F,\lambda)| = 0 \tag{A.16}$$

Using the above algebraic theory, Eqs. (A.10) and (A.12) can be written as follows:

$$\left|\sum M\right| + \left|\sum (Q_i \cdot \cos\theta_i)\right| = \left|\sum (Q_i \cdot \cos\theta_i) \cdot Y_i + Q_i \cdot \sin\theta_i \cdot X_i)\right| + \left|\sum (Q_i \cdot \cos\theta_i)\right| = 0 \tag{A.17}$$

Eq. (A.17) is now solved using the genetic algorithm that is explained in Section 3.1.

References

[1] Baker R. Determination of critical slip surface in slope stability computations. Int J Numer Anal Meth Geomech 1980;4:333–59.

[2] Bardet JP, Kapuskar MM. A simplex analysis of slope stability. Comput Geotech 1989;8:329–48.

[3] Bishop AW. The use of the slip circle in the stability analysis of earth slopes. Geotechnique 1955;5(1):7–17.

[4] Bolton Hermanus PJ, Heymann G, Groenwold A. Global search for critical failure surface in slope stability analysis. Eng Opt 2003;35(1):51–65.

[5] Celestino TB, Duncan JM. Simplified search for non-circular slip surface. In: Proceedings of the 10th international conference on soil mechanics and foundation engineering; 1981.

[6] Goldberg DE. Genetic algorithms in search, optimisation, and machine learning. Addison-Wesley; 1989.

[7] Holland J. Adaptation in natural and artificial systems. University of Michigan Press; 1975.

[8] Li KS, White W. Rapid evaluation of the critical slip surface in slope stability problems. Int J Numer Anal Meth Geomech 1987;11:440–73.
[9] McCombie P, Wilkinson P. The use of the simple genetic algorithm in finding the critical factor of safety in slope stability analysis. Comput Geotech 2002;29:699–714.
[10] Morgenstern NR, Price VE. The analysis of the stability of general slip surfaces. Geotechnique 1965;15:79–93.
[11] Nguyen VU. Determination of critical slope failure surface. J Geotech Eng 1985;111(2):238–50.
[12] Spencer E. A method of analysis for stability of embankments using parallel inter-slice forces. Geotechnique 1967; 17:11–26.
[13] Zhu DY, Lee CF, Qian QH, Zou ZS, Sun F. A new procedure for computing the factor of safety using the Morgenstern–Price method. Can Geotech J 2001;38(4):882–8.