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The planar hub location problem: a probabilistic clustering approach

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Abstract Given the demand between each origin-destination pair on a network, the planar hub location problem is to locate the multiple hubs anywhere on the plane and to assign the traffic to them so as to minimize the total travelling cost. The trips between any two points can be nonstop (no hubs used) or started by visiting any of the hubs. The travel cost between hubs is discounted with a factor. It is assumed that each point can be served by multiple hubs.

We propose a probabilistic clustering method for the planar hub-location problem which is analogous to the method of Iyigun and Ben-Israel (in Operations Research Letters 38, 207–214, 2010; Computational Optmization and Applications, 2013) for the solution of the multi-facility location problem. The proposed method is an iterative probabilistic approach assuming that all trips can be taken with probabilities that depend on the travel costs based on the hub locations. Each hub location is the convex combination of all data points and other hubs. The probabilities are updated at each iteration together with the hub locations. Computations stop when the hub locations stop moving.

Fermat-Weber problem and multi-facility location problem are the special cases of the proposed approach.

Keywords Hub location problem · Planar hub location · Clustering · Fermat–Weber problem · Probabilistic assignments

1 Introduction

We consider a transportation network consisting of N *cities*, with known *locations* $\{ \mathbf{x}_i : i \in \overline{1, N}\}$ and known *demands* for travel between cities, $w_{ij} = \{$ the demand for travel from city i to city $j\}, \quad i, j \in \overline{1, N}$.

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To accommodate all traffic and make it more efficient, some (or all) of the traffic is directed through hubs. A *hub* is a facility where passengers from several nearby origins can be pooled for a trip to a common destination, or to another hub (from where the passengers continue to their destinations). By combining trips and directing them through hubs, the sum of distances traveled in the system can be reduced. Another advantage is greater efficiency of travel, because typically bigger planes are used between hubs, and they are flown at higher altitudes.

The *hub location problem* (HLP) is to locate K hubs in the network so as to minimize the total travel cost in the system.

In some HLP cases the hub locations are constrained to lie in a given subset of the plane, in particular a given subset of the data points. This constrained problem is called *discrete hub location* model. It was first considered by O'Kelly (1987), introducing a quadratic integer program for location of interacting hub facilities. Alternative integer linear programming formulations of discrete hub location problems have also been provided by Campbell (1994a), Bryan (1998), Ernst and Krishnamoorthy (1996, 1998), and O'Kelly et al. (1996).

The original formulation in O'Kelly (1987) assumes a *single assignment* from a datapoint to a unique hub. In some of the other HLP models, it is allowed or required, for a customer to be connected to more than one hub (called *multiple-assignment*). In general, discrete HLP is often solved by integer programming (i.e. Labbé et al. 2005). Although most of the single assignment problems are addressed by heuristics (see, Ernst and Krishnamoorthy 1996; Ernst et al. 2002 and Gavriliouk 2009), integer programming models are studied for solving multiple assignment discrete HLP problems with small gaps of the corresponding linear programming relaxations (Hamacher et al. 2004). In recent studies, Bender's decomposition is successfully used for solving large problems (see Camargo et al. 2008 and Contreras et al. 2011).

In the *planar* or *continuous* version of HLP, the hubs can be located anywhere in the plane. It is originally considered in O'Kelly (1986) and subsequently the problem was studied by Aykin (1988, 1995), and Aykin and Brown (1992). A clustering approach is presented in O'Kelly (1992) for solving the planar HLP. Although several computational approaches are studied for the discrete hub location problem, the computational developments for the planar case is limited in the literature (see Campbell 1994b; Bryan and O'Kelly 1999; Alumur and Kara 2008 and Campbell and O'Kelly 2012 for detailed reviews).

We study here the *continuous*, or *planar*, HLP model and it is assumed that the data points (customers) can be served by multiple hubs. We propose a probabilistic approximation of HLP, analogous to the method proposed in Iyigun and Ben-Israel (2010), Iyigun and BenIsrael (2013) for the solution of the multi-facility location problem, see Sect. 2.

The major contribution of this paper is to propose an approach for solving planar HLP. The proposed method is quite efficient for solving large instances of planar HLPs which is not possible in the discrete case. The approach is originating from a clustering method and it enables to solve large instances of continuous version of HLP. Many real world problems from postal and trucking logistics, aerial transportation, and telecommunication applications motivate solving large scale instances of continuous hub location problem.

The plan of the paper is as follows. Section 2 describes the multi-facility location problem and Sect. 3 defines the terminology used and describes the hub location problem (H.K). In Sect. 4, calculation of travel costs is explained.

Section 5 introduces probabilistic assignments of trips, with cluster probabilities that depend on the trip costs. Section 6 introduces the probabilistic hub location problem (HP.K), an approximation of (H.K). The center updates of (HP.K) are explained in Sect. 7 and the proposed algorithm is given in Sect. 8. The paper is concluded with the illustration of the proposed approach, in Sect. 9.

2 The multi facility location problem

We denote an index set $\{1,\ldots,K\}$ by $\overline{1,K}$. For a vector $\mathbf{x} = (x_j) \in \mathbb{R}^n$, $\|\mathbf{x}\|$ denotes the Euclidean norm,

$$\|\mathbf{x}\| := \sqrt{\sum_{j=1}^{n} |x_j|^2}. \tag{1}$$

The Euclidean distance $d(\mathbf{x},\mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$ is used throughout.

Given

- positive integers n, N,
- $\mathbf{X} = \{\mathbf{x}_i : i \in \overline{1,N}\}$ a set of N points (*customers*) in $\mathbb{R}^n$,
- $\mathbf{w} = \{w_i : i \in \overline{1,N}\}$ a set of corresponding N positive weights (*demands*) $w_i > 0$, and
- an integer K, $1 \le K \le N$,

the *multi-facility location problem* (MFLP) is to locate K points (*centers* or *facilities*) $\{\mathbf{c}_k : k \in \overline{1,K}\}$ in $\mathbb{R}^n$, and to assign each customer to a center, so as to minimize the weighted sum of distances traveled

$$\min_{\mathbf{c}_1,\mathbf{c}_2,\ldots,\mathbf{c}_K} \sum_{k=1}^{K} \sum_{\mathbf{x}_i \in \mathbf{X}_k} w_i\, d(\mathbf{x}_i, \mathbf{c}_k) \tag{L.K}$$

where $\mathbf{X}_k$ is the cluster of customers assigned to the kth facility. The case K = N (every point is a center), is of no interest. The case K = 1 is the *Fermat–Weber location problem* (Drezner et al. 2002), where assignment is absent,

$$\min_{\mathbf{c}} \sum_{i=1}^{N} w_i\, d(\mathbf{x}_i, \mathbf{c}). \tag{L.1}$$

The objective function of (L.1)

$$f(\mathbf{c}) = \sum_{i=1}^{N} w_i\, d(\mathbf{x}_i, \mathbf{c}) \tag{2}$$

is convex (strictly convex if the data points are not collinear), and its gradient

$$\nabla f(\mathbf{c}) = -\sum_{i=1}^{N} w_i \frac{\mathbf{x}_i - \mathbf{c}}{\|\mathbf{x}_i - \mathbf{c}\|} \tag{3}$$

exists for $\mathbf{c} \in \mathbf{X}$, i.e., $\mathbf{c} = \mathbf{x}_i, i \in \overline{1, N}$. The optimality condition $\nabla f(\mathbf{c}^*) = \mathbf{0}$ gives the optimal center $\mathbf{c}^*$ as a convex combination of the points $\{\mathbf{x}_i : i \in \overline{1,N}\}$,

$$\mathbf{c}^* = \sum_{i=1}^{N} \lambda_i(\mathbf{c}^*)\mathbf{x}_i, \quad \lambda_i(\mathbf{c}^*) = \frac{w_i/\|\mathbf{x}_i - \mathbf{c}^*\|}{\sum_{j=1}^{N} w_j/\|\mathbf{x}_j - \mathbf{c}^*\|}, \tag{4}$$

with weights depending on $\mathbf{c}^*$, giving rise to the *Weiszfeld iteration* (Weiszfeld 1937), for the updated center $\mathbf{c}_+$ in terms of the current center $\mathbf{c}$,

$$\mathbf{c}_+ := \sum_{i=1}^{N} \left(\frac{w_i/\|\mathbf{x}_i - \mathbf{c}\|}{\sum_{m=1}^{N} w_m/\|\mathbf{x}_m - \mathbf{c}\|} \right) \mathbf{x}_i, \quad \text{if } \mathbf{c} \in \mathbf{X}, \tag{5}$$

with some modification for points $\mathbf{c} \in \mathbf{X}$ where $\nabla f(\mathbf{c})$ is undefined (Vardi and Zhang 2001).

For $1 < K < N$, the problem (L.K) is NP hard (Megiddo and Supowit 1984). It can be solved polynomially in N for K = 2, see Drezner (1984), and possibly for other given K. A heuristic method given in Iyigun and Ben-Israel (2010, 2013) replaces the rigid assignments of points $\{\mathbf{x}_i\}$ to the clusters $\{\mathbf{X}_k\}$ by membership probabilities,

$$p_k(\mathbf{x}_i) = \text{Prob}\{\mathbf{x}_j \in \mathbf{X}_k\}, \quad i \in \overline{1, N}, k \in \overline{1, K}, \tag{6}$$

assumed to depend on the distances $\{d(\mathbf{x}_i, \mathbf{c}_k)\}$. The combinatorial problem (L.K) is approximated by the probabilistic problem

$$\min_{\substack{\{\mathbf{c}_1,\dots,\mathbf{c}_K\} \\ \{\{p_1(\mathbf{x}_i),\dots,p_K(\mathbf{x}_i)\}: i \in \overline{1,N}\}}} \sum_{k=1}^{K} \sum_{i=1}^{N} w_i \, p_k(\mathbf{x}_i) \, d(\mathbf{x}_i, \mathbf{c}_k), \tag{P.K}$$

with two sets of variables, the *centers* $\{\mathbf{c}_k\}$ and *probabilistic assignments* $\{p_k(\mathbf{x}_i)\}$, that are updated iteratively. The problem (P.K) uses the same data as problem (L.K).

The problem (P.K) separates into K single facility location problems, coupled by the probabilities $\{p_k(\mathbf{x}_i)\}$. Indeed, for fixed probabilities $\{p_k(\mathbf{x}_i)\}$, the objective function of (P.K) is a separable function of the K centers

$$f(\mathbf{c}_1, \ldots, \mathbf{c}_K) = \sum_{k=1}^{K} f_k(\mathbf{c}_k), \quad \text{where} \quad f_k(\mathbf{c}) = \sum_{k=1}^{K} \sum_{i=1}^{N} w_i \, p_k(\mathbf{x}_i) \, d(\mathbf{x}_i, \mathbf{c}), k \in \overline{1, K}, \tag{7}$$

and each $f_k(\mathbf{c})$ can be minimized separately.

3 The problem

Given

- positive integers n,N,
- $\mathbf{X} = \{\mathbf{x}_i : i \in \overline{1, N}\}$ a set of N points (*cities*) in R^n,
- $\mathbf{W} = \{ w_{ij} : i,j \in 1,N \}$ a set of corresponding N^2 positive weights (*demands*) $w_{ij} \geq 0$, and
- an integer K, $1 \leq K \leq N$,

the *hub location problem* (HLP) is to locate K hubs $\{\mathbf{c}_k : k \in \overline{1, K}\}$, so as to minimize the total travel costs in the system,

$$\min_{\{\mathbf{c}_1,\ldots,\mathbf{c}_K\}} \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij} c(\mathbf{x}_i, \mathbf{x}_j), \tag{H.K}$$

where $c(\mathbf{x}_i,\mathbf{x}_j)$ is the minimal cost of travel from $\mathbf{x}_i$ to $\mathbf{x}_j$, a cost that depends on the hub locations and usage, see (15) below.

We assume

$$\boxed{\text{there is a direct connection between any city and any hub}} \tag{8}$$

and

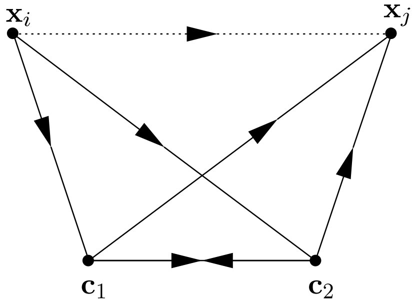


Fig. 1 Illustration of Example 1

$$\boxed{\text{travel between any two cities may use at most two hubs.}} \tag{9}$$

Some terminology: A *route* is any path between 2 cities, directly or through (at most two) hubs. Given K hubs, the number of routes from $\mathbf{x}_i$ to $\mathbf{x}_j$ (or any two other cities) is $1 + K + K(K-1) = K^2 + 1$. For any hub $\mathbf{c}_k$, let $\mathbf{R}_k(\mathbf{x}_i,\mathbf{x}_j)$ denote the set of routes from $\mathbf{x}_i$ to $\mathbf{x}_j$ with $\mathbf{x}_i \rightarrow \mathbf{c}_k$ as the first *stop*, and let $\mathbf{R}_0(\mathbf{x}_i,\mathbf{x}_j)$ denote the *nonstop* (direct) route (not using hubs). The cheapest route (see discussion of costs in Sect. 4) in $\mathbf{R}_k(\mathbf{x}_i,\mathbf{x}_j)$ is called the kth *trip* from $\mathbf{x}_i$ to $\mathbf{x}_j$, $k \in \overline{0, K}$.

In a network with N cities and K hubs, there are $2*\binom{N}{2}(K^2+1)$ routes, and $2*\binom{N}{2}(K+1)$ trips.

Example 1 Given 2 hubs, $\mathbf{c}_1$ and $\mathbf{c}_2$, and any two cities $\mathbf{x}_i$, $\mathbf{x}_j$, there are 5 possible routes from $\mathbf{x}_i$ to $\mathbf{x}_j$, see Fig. 1,

R_1: $\mathbf{x}_i \rightarrow \mathbf{x}_j$

R_2: $\mathbf{x}_i \rightarrow \mathbf{c}_1 \rightarrow \mathbf{x}_j$

R_3: $\mathbf{x}_i \rightarrow \mathbf{c}_1 \rightarrow \mathbf{c}_2 \rightarrow \mathbf{x}_j$

R_4: $\mathbf{x}_i \rightarrow \mathbf{c}_2 \rightarrow \mathbf{x}_j$

R_5: $\mathbf{x}_i \rightarrow \mathbf{c}_2 \rightarrow \mathbf{c}_1 \rightarrow \mathbf{x}_j$

There are 3 trips, the cheapest routes in the sets $\mathbf{R}_0(\mathbf{x}_i,\mathbf{x}_j) = \{R_1\}$, $\mathbf{R}_1(\mathbf{x}_i,\mathbf{x}_j) = \{R_2, R_3\}$ and $\mathbf{R}_2(\mathbf{x}_i,\mathbf{x}_j) = \{R_4, R_5\}$. In particular, R_1 is a trip, the 0th trip from $\mathbf{x}_i$ to $\mathbf{x}_j$.

4 Costs

Nonstop travel between cities The cost of traveling directly from city i to city j is denoted $c_0(\mathbf{x}_i,\mathbf{x}_j)$, and is proportional to the Euclidean distance between $\mathbf{x}_i$ and $\mathbf{x}_j$ (if there is a direct connection), and can be identified with it

$$c_0(\mathbf{x}_i,\mathbf{x}_j) = \begin{cases} d(\mathbf{x}_i,\mathbf{x}_j), & \text{if the cities are connected}, \\ \infty & \text{otherwise}, \end{cases} \quad \forall i,j \in \overline{1,N}. \tag{10}$$

Direct travel between hubs We denote the location of the kth-hub by $\mathbf{c}_k$, $k \in \overline{1,K}$. The cost of traveling directly from the hub $\mathbf{c}_k$ to the hub $\mathbf{c}_\ell$, $c_0(\mathbf{c}_k,\mathbf{c}_\ell)$, is proportional to the Euclidean distance between $\mathbf{c}_k$ and $\mathbf{c}_\ell$, with a discount factor $\alpha(k,\ell)$,

$$c_0(\mathbf{c}_k,\mathbf{c}_\ell) = \alpha(k,\ell)\, d(\mathbf{c}_k,\mathbf{c}_\ell), \quad \forall k,\ell \in \overline{1,K}, \tag{11}$$

where $0 < \alpha(k,\ell) \le 1$. Here $1 - \alpha(k,\ell)$ is the cost saving per mile traveled between these two specific hubs.

Direct travel between cities and hubs No savings are assumed for a direct travel between cities and hubs, and the cost is therefore

$$c_0(\mathbf{x}_i,\mathbf{c}_k) = c_0(\mathbf{c}_k,\mathbf{x}_i) = d(\mathbf{x}_i,\mathbf{c}_k), \quad \forall i \in \overline{1,N},\ k \in \overline{1,K}. \tag{12}$$

Minimal cost of routes from hubs to cities The minimal cost among all routes connecting hub $\mathbf{c}_k$ and city $\mathbf{x}_j$ is, by (9),

$$c(\mathbf{c}_k,\mathbf{x}_j) = \min\left\{d(\mathbf{c}_k,\mathbf{x}_j),\ \min_{k\neq\ell\in 1:K}\{\alpha(k,\ell)\, d(\mathbf{c}_k,\mathbf{c}_\ell) + d(\mathbf{c}_\ell,\mathbf{x}_j)\}\right\}. \tag{13}$$

Costsoftrips Recall that a trip is the cheapest route with a specified first stop. For $k \in \overline{0,K}$, the cost of the kth trip from $\mathbf{x}_i$ to $\mathbf{x}_j$ is, by (10) and (13),

$$c_k(\mathbf{x}_i,\mathbf{x}_j) = \begin{cases} \text{from (10)}, & k = 0 \\ d(\mathbf{x}_i,\mathbf{c}_k) + c(\mathbf{c}_k,\mathbf{x}_j), & k \in \overline{1,K} \end{cases}; \tag{14}$$

Minimal travel costs Finally, the *minimal cost* of travel from city $\mathbf{x}_i$ to city $\mathbf{x}_j$, denoted $c(\mathbf{x}_i,\mathbf{x}_j)$, is given by (14),

$$c(\mathbf{x}_i,\mathbf{x}_j) = \min\{c_k(\mathbf{x}_i,\mathbf{x}_j) : k \in \overline{0,K}\}. \tag{15}$$

The two extreme cases for the discount functions in (11) are $\alpha(k,\ell) =$ α(k,) 1 (i.e. no advantage to travel between hubs $\mathbf{c}_k$ and $\mathbf{c}_\ell$) and $\alpha(k,\ell) = 0$, where the travel between these hubs is free. If α(k,) = 1 for all pairs of hubs, we expect all hubs to collapse into one hub, see Corollary 1 below. If $\alpha(k,\ell) = 0$ for all pairs of hubs, we expect the hubs to well separated, and to serve their regions, because then only the local travel matters.

Corollary 1 *If* α(k,) = 1 *for all pairs of hubs* $(\mathbf{c}_k,\mathbf{c}_\ell)$ *then the hub location problem* (H.K) *reduces to the Fermat–Weber location problem* (L.1) *of finding the center of all cities* $\mathbf{x}_i$ *with corresponding weights*

$$w_i = \sum\{w_{ij} : j \in \overline{1,N}, \text{ there is no direct connection from } \mathbf{x}_i \text{ to } \mathbf{x}_j\}, \quad i \in \overline{1,N}. \tag{16}$$

Proof Let α(k,) = 1 for all pairs of hubs. If there is a direct connection from $\mathbf{x}_i$ to $\mathbf{x}_j$ then by the triangular inequality the travel from $\mathbf{x}_i$ to $\mathbf{x}_j$ will not use a hub. If there is no direct connection then, by the triangular inequality again, only one hub will be used. It follows from (8) that all hubs collapse to one, the center of the points $\{\mathbf{x}_i : i \in \overline{1,N}\}$ where the weight of each point is the sum of the demands w_{ij} that cannot be shipped directly.

5 Trip probabilities

Given the hubs and their locations, all trips in the system can be determined by (15), finding the optimal routes between any two cities directly or through intermediate hubs. The traffic patterns can then be used to update the hub locations, and the trips are calculated again, etc.

We propose an alternative, probabilistic approach, as in Iyigun and Ben-Israel (2013), by assuming that all trips can be taken with probabilities that depend on the travel costs. For any two cities $\mathbf{x}_i, \mathbf{x}_j$ and $k \in \overline{0,K}$, let $p_k(i,j)$ denote the probability of taking the kth trip from $\mathbf{x}_i$ to $\mathbf{x}_j$. In particular, $p_0(i,j)$ is the probability of direct travel from $\mathbf{x}_i$ to $\mathbf{x}_j$.

As in Iyigun and Ben-Israel (2013) we assume the principle

$$\boxed{\text{a trip is more likely the lower its cost}} \tag{A}$$

which we model, for any pair of cities $\mathbf{x}_i$ and $\mathbf{x}_j$, by the equations,

$$p_k(i,j)c_k(\mathbf{x}_i,\mathbf{x}_j) = \frac{1}{w_{ij}} C(\mathbf{x}_i,\mathbf{x}_j), \quad k \in \overline{0,K}, \tag{17}$$

where $C(\mathbf{x}_i,\mathbf{x}_j)$ is a function of $(\mathbf{x}_i,\mathbf{x}_j)$, independent of k. The function $C(\mathbf{x}_i,\mathbf{x}_j)$ is called the joint cost function of the pair $(\mathbf{x}_i,\mathbf{x}_j)$. It is analogous to the joint distance function introduced in Ben-Israel and Iyigun (2008).

Using the fact that probabilities add to one, we get from (17),

$$p_k(i,j) = \frac{1/c_k(\mathbf{x}_i,\mathbf{x}_j)}{\sum_{\ell=0}^{K} 1/c_\ell(\mathbf{x}_i,\mathbf{x}_j)} = \frac{\prod_{t\neq k} c_t(\mathbf{x}_i,\mathbf{x}_j)}{\sum_{\ell=0}^{K} \prod_{m\neq\ell} c_m(\mathbf{x}_i,\mathbf{x}_j)}, \quad k \in \overline{0,K}, \tag{18}$$

and the joint cost function,

$$C(\mathbf{x}_i,\mathbf{x}_j) = w_{ij} \frac{\prod_{k=0}^{K} c_k(\mathbf{x}_i,\mathbf{x}_j)}{\sum_{\ell=0}^{K} \prod_{m\neq\ell} c_m(\mathbf{x}_i,\mathbf{x}_j)}, \tag{19}$$

which is, up to a constant, the harmonic mean of the K + 1 trip costs. $\{c_k(\mathbf{x}_i,\mathbf{x}_j) : k \in \overline{0,K}\}$

In the special case K = 2,

$$p_0(i,j) = \frac{c_1(\mathbf{x}_i,\mathbf{x}_j)c_2(\mathbf{x}_i,\mathbf{x}_j)}{c_0(\mathbf{x}_i,\mathbf{x}_j)c_1(\mathbf{x}_i,\mathbf{x}_j) + c_0(\mathbf{x}_i,\mathbf{x}_j)c_2(\mathbf{x}_i,\mathbf{x}_j) + c_1(\mathbf{x}_i,\mathbf{x}_j)c_2(\mathbf{x}_i,\mathbf{x}_j)}$$

$$p_1(i,j) = \frac{c_0(\mathbf{x}_i,\mathbf{x}_j)c_2(\mathbf{x}_i,\mathbf{x}_j)}{c_0(\mathbf{x}_i,\mathbf{x}_j)c_1(\mathbf{x}_i,\mathbf{x}_j) + c_0(\mathbf{x}_i,\mathbf{x}_j)c_2(\mathbf{x}_i,\mathbf{x}_j) + c_1(\mathbf{x}_i,\mathbf{x}_j)c_2(\mathbf{x}_i,\mathbf{x}_j)},$$

$$p_2(i,j) = \frac{c_0(\mathbf{x}_i,\mathbf{x}_j)c_1(\mathbf{x}_i,\mathbf{x}_j)}{c_0(\mathbf{x}_i,\mathbf{x}_j)c_1(\mathbf{x}_i,\mathbf{x}_j) + c_0(\mathbf{x}_i,\mathbf{x}_j)c_2(\mathbf{x}_i,\mathbf{x}_j) + c_1(\mathbf{x}_i,\mathbf{x}_j)c_2(\mathbf{x}_i,\mathbf{x}_j)},$$

and,

$$C(\mathbf{x}_i,\mathbf{x}_j) = w_{ij} \frac{c_1(\mathbf{x}_i,\mathbf{x}_j)c_2(\mathbf{x}_i,\mathbf{x}_j)c_3(\mathbf{x}_i,\mathbf{x}_j)}{c_0(\mathbf{x}_i,\mathbf{x}_j)c_1(\mathbf{x}_i,\mathbf{x}_j) + c_0(\mathbf{x}_i,\mathbf{x}_j)c_2(\mathbf{x}_i,\mathbf{x}_j) + c_1(\mathbf{x}_i,\mathbf{x}_j)c_2(\mathbf{x}_i,\mathbf{x}_j)}.$$

6 An extremum problem

Abbreviating $p_k(i,j)$ by p_k, Eqs. (17) are an optimality condition for the extremum problem

$$\min \left\{ w_{ij} \sum_{k=0}^{K} p_k^2 c_k(\mathbf{x}_i,\mathbf{x}_j) : \sum_{k=0}^{K} p_k = 1,\ p_k \geq 0,\ k \in \overline{0,K} \right\} \tag{20}$$

with variables $\{p_k\}$. The squares of probabilities in (20) are explained as a device for smoothing the underlying objective, $\min\{c_k(\mathbf{x}_i,\mathbf{x}_j) : k \in \overline{0,K}\}$, see the seminal paper by Teboulle (2007).

Recall the *hub location problem*: given integers $1 \leq K < N$, a set of N cities $\{i : i \in \overline{1,N}\}$, their locations $\{\mathbf{x}_i\}$, and N^2 demands $\{w_{ij}\}$, determine the locations $\{\mathbf{c}_k : k \in \overline{1,K}\}$ of K hubs, so as to minimize the sum of costs of travel,

$$\min \sum_{\{i,j\} \in \overline{1,N}} w_{ij} c(\mathbf{x}_i,\mathbf{x}_j) \tag{21}$$

with $c(\mathbf{x}_i,\mathbf{x}_j)$ as in (15).

The hub location problem (21) can thus be approximated, using (20), by the minimization problem

$$\min \sum_{k=0}^{K} \sum_{\substack{\{i,j\}\in\overline{1,N}\\ i\neq j}} w_{ij}\, p_k^2(i,j)\, c_k(\mathbf{x}_i,\mathbf{x}_j)$$

(HP.K) s.t.

$$\sum_{k=0}^{K} p_k(i,j) = 1, \quad i,j \in \overline{1,N},$$

$$p_k(i,j) \geq 0, \quad k \in \overline{0,K},\ i,j \in \overline{1,N},$$

with two sets of variables, the *hub locations* $\{\mathbf{c}_1,\ldots,\mathbf{c}_K\}$ and *probabilities* $\{p_k(i,j) : k \in \overline{0,K}, (i,j) \in \overline{1,N}\}$, corresponding, respectively, to the centers and assignments of the original problem (21).

7 Probabilities and centers

The objective function of (HP.K) is denoted

$$f(\mathbf{c}_1,\ldots,\mathbf{c}_K) := \sum_{k=0}^{K} \sum_{\substack{\{i,j\}\in\overline{1,N}\\ i\neq j}} p_k^2(i,j)\, w_{ij}\, c_k(\mathbf{x}_i,\mathbf{x}_j). \tag{22}$$

A natural approach to solving (HP.K), see e.g. Cooper (1964), is to fix one set of variables, and minimize (HP.K) with respect to the other set, then fix the other set, etc. We thus alternate between

(1) the *probabilities problem*, i.e. (HP.K) with given hub locations, and

(2) the *centers problem*, (HP.K) with given assignment probabilities, and

update their solutions as follows:

Probabilities update With the hub locations given, the distances $d(\mathbf{x}_i,\mathbf{c}_k)$ computed for all hub locations $\mathbf{c}_k$ and data points $\mathbf{x}_i$, and the distances between data points $\mathbf{x}_i$, $\mathbf{x}_j$, the minimizing probabilities are given explicitly by (18),

$$p_k(i,j) = \frac{1/c_k(\mathbf{x}_i,\mathbf{x}_j)}{\sum_{\ell=1}^{K} 1/c_\ell(\mathbf{x}_i,\mathbf{x}_j)} = \frac{\prod_{t\neq k} c_t(\mathbf{x}_i,\mathbf{x}_j)}{\sum_{\ell=1}^{K} \prod_{m\neq\ell} c_m(\mathbf{x}_i,\mathbf{x}_j)}, \quad k \in \overline{0,K}. \tag{23}$$

Centers update Fixing all the probabilities $p_k(i,j)$ in (HP.K), the objective function (22) is a non-separable function of the hubs. The kth hub $\mathbf{c}_k$ appears in all the kth trips ($\mathbf{x}_i \to \mathbf{c}_k \to \cdots$), but may also appear in some of the th trips ($\cdots \to \mathbf{c}_\ell \to \mathbf{c}_k \to \mathbf{x}_j$), $i,j \in \overline{1,N}$.

In general, a hub may appear in all trips except for the non-stop trips. Taking the partial derivatives of (22) with respect to $\mathbf{c}_k$ gives the hub $\mathbf{c}_k$ as a convex combination of the data points $\mathbf{x}_i$, and of other hubs $\mathbf{c}$ that communicate with it.

This is different than in the problem (P.K), where the objective (7) is a separable function of the centers, that can be solved separately, with each center a convex combination of the data points.

For any pair of cities $i, j \in \overline{1, N}$, $i \neq j$, and a hub $k \in \overline{1, 2}$, define the functions,

$$\delta_{k\ell}(i,j) = \begin{cases} 1, & \text{if the trip from city i to city j first visits hub k, through hub } \ell, \text{ if } \ell \neq 0, \text{ and goes directly to city } j \text{ if } \ell = 0, \\ 0 & \text{otherwise.} \end{cases} \quad \ell \in \overline{0,2},\ \ell \neq k. \tag{24}$$

Then the cost of travel in (14) can be written as

$$c_k(\mathbf{x}_i, \mathbf{x}_j) = d(\mathbf{x}_i, \mathbf{c}_k) + \sum_{\substack{\ell=0 \\ \ell \neq k}}^{K} \delta_{k\ell}(i, j) c_\ell(\mathbf{c}_k, \mathbf{x}_j) \tag{25}$$

where $\sum_{\substack{\ell=0 \\ \ell \neq k}}^{K} \delta_{k\ell}(i, j) = 1$.

The cost term in (25), $c_\ell(\mathbf{c}_k, \mathbf{x}_j)$ will be

$$c_0(\mathbf{c}_k, \mathbf{x}_j) = d(\mathbf{c}_k, \mathbf{x}_j), \quad \text{if } \ell = 0;$$

$$c_\ell(\mathbf{c}_k, \mathbf{x}_j) = \alpha(k, \ell)\, d(\mathbf{c}_k, \mathbf{c}_\ell) + d(\mathbf{c}_\ell, \mathbf{x}_j), \quad \text{if } \ell \neq 0.$$

For simplicity consider the case of two hubs (the results are easily extended to the general case). Assume that the probabilities $p_0(i,j)$, $p_1(i,j)$ and $p_2(i,j)$ are given for $\{i, j\} \in \overline{1, N}$. The objective in (22) is

$$f(\mathbf{c}_1, \mathbf{c}_2) = \sum_{\substack{\{i,j\} \in \overline{1,N} \\ i \neq j}} w_{ij} \big(p_0(i, j)^2 c_0(\mathbf{x}_i, \mathbf{x}_j) + p_1(i, j)^2 c_1(\mathbf{x}_i, \mathbf{x}_j) + p_2(i, j)^2 c_2(\mathbf{x}_i, \mathbf{x}_j) \big) \tag{26}$$

where (by using the functions (24)),

$$c_1(\mathbf{x}_i, \mathbf{x}_j) = d(\mathbf{x}_i, \mathbf{c}_1) + \delta_{10}(i, j)\, d(\mathbf{c}_1, \mathbf{x}_j) + \delta_{12}(i, j) \big(\alpha(1, 2) d(\mathbf{c}_1, \mathbf{c}_2) + d(\mathbf{c}_2, \mathbf{x}_j) \big) \tag{27}$$

$$c_2(\mathbf{x}_i, \mathbf{x}_j) = d(\mathbf{x}_i, \mathbf{c}_2) + \delta_{20}(i, j)\, d(\mathbf{c}_2, \mathbf{x}_j) + \delta_{21}(i, j) \big(\alpha(2, 1) d(\mathbf{c}_2, \mathbf{c}_1) + d(\mathbf{c}_1, \mathbf{x}_j) \big). \tag{28}$$

Theorem 1 *Let the distance functions*, $d(\mathbf{c}_k,\mathbf{c}_l)$ *in* (11) *and* $d(\mathbf{x}_i,\mathbf{c}_k)$ *in* (12) *be Euclidean, and the cost functions* $c_k(\mathbf{x}_i,\mathbf{x}_j)$ *be computed as in* (14). *Use the decision functions*, δ_k *in* (24).

Then the minimizers $\mathbf{c}_1,\mathbf{c}_2$ *of* (26), *if they do not coincide with any of the points* $\mathbf{x}_i$, $i \in \overline{1,N}$, *are given by*

$$\mathbf{c}_1 = \sum_{\{i,j\}\in\overline{1,N}} \frac{\lambda_{1i}^{(i,j)}\mathbf{x}_i}{\Lambda_1} + \sum_{\{i,j\}\in\overline{1,N}} \frac{\lambda_{1j}^{(i,j)}\mathbf{x}_j}{\Lambda_1} + \sum_{\{i,j\}\in\overline{1,N}} \frac{\mu_{12}^{(i,j)}\mathbf{c}_2}{\Lambda_1} \tag{29}$$

a convex combination of the points $\{\mathbf{x}_i\}$ *and the other hub, where,*

$$\lambda_{1i}^{(i,j)} = \frac{w_{ij}\, p_1(i,j)^2}{d(\mathbf{x}_i,\mathbf{c}_1)}, \tag{30}$$

$$\lambda_{1j}^{(i,j)} = \frac{w_{ij}(\delta_{10}(i,j)\, p_1(i,j)^2 + \delta_{21}(i,j)\, p_2(i,j)^2)}{d(\mathbf{c}_1,\mathbf{x}_j)}, \tag{31}$$

$$\mu_{12}^{(i,j)} = \frac{\alpha(1,2)\, w_{ij}(\delta_{12}(i,j)\, p_1(i,j)^2 + \delta_{21}(i,j)\, p_2(i,j)^2)}{d(\mathbf{c}_1,\mathbf{c}_2)}, \tag{32}$$

and

$$\Lambda_1 = \sum_{\{i,j\}\in\overline{1,N}} \left(\lambda_{1i}^{(i,j)} + \lambda_{1j}^{(i,j)} + \mu_{12}^{(i,j)}\right). \tag{33}$$

Similarly,

$$\mathbf{c}_2 = \sum_{\substack{\{i,j\}\in\overline{1,N}\\ i\neq j}} \frac{\lambda_{2i}^{(i,j)}\mathbf{x}_i}{\Lambda_2} + \sum_{\substack{\{i,j\}\in\overline{1,N}\\ i\neq j}} \frac{\lambda_{2j}^{(i,j)}\mathbf{x}_j}{\Lambda_2} + \sum_{\substack{\{i,j\}\in\overline{1,N}\\ i\neq j}} \frac{\mu_{21}^{(i,j)}\mathbf{c}_1}{\Lambda_2} \tag{34}$$

where

$$\lambda_{2i}^{(i,j)} = \frac{w_{ij}\, p_2(i,j)^2}{d(\mathbf{x}_i,\mathbf{c}_2)},$$

$$\lambda_{2j}^{(i,j)} = \frac{w_{ij}(\delta_{20}(i,j)\, p_2(i,j)^2 + \delta_{12}(i,j)\, p_1(i,j)^2)}{d(\mathbf{c}_2,\mathbf{x}_j)},$$

$$\mu_{21}^{(i,j)} = \frac{\alpha(2,1)\, w_{ij}(\delta_{21}(i,j)\, p_2(i,j)^2 + \delta_{12}(i,j)\, p_1(i,j)^2)}{d(\mathbf{c}_2,\mathbf{c}_1)},$$

and

$$\Lambda_2 = \sum \left(\lambda_{2i}^{(i,j)} + \lambda_{2j}^{(i,j)} + \mu_{21}^{(i,j)}\right).$$

Proof The gradient of $d(\mathbf{x},\mathbf{c}) = \|\mathbf{x}-\mathbf{c}\|$ with respect to $\mathbf{c}$ is, for $\mathbf{x} \neq \mathbf{c}$,

$$\nabla_{\mathbf{c}} \|\mathbf{x}-\mathbf{c}\| = -\frac{\mathbf{x}-\mathbf{c}}{\|\mathbf{x}-\mathbf{c}\|} = -\frac{\mathbf{x}-\mathbf{c}}{d(\mathbf{x},\mathbf{c})}. \tag{35}$$

Substitute (27) and (28) in the objective function of (26) for the cost terms c_1, c_2 and the gradient of (26) with respect to $\mathbf{c}_1$ is

$$\begin{aligned}
\nabla_{\mathbf{c}_1} f(\mathbf{c}_1,\mathbf{c}_2) = &\sum_{\substack{\{i,j\}\in\overline{1,N}\\ i\neq j}} w_{ij}\Bigg[p_1^2(i,j)\Bigg(-\frac{\mathbf{x}_i-\mathbf{c}_1}{d(\mathbf{x}_i,\mathbf{c}_1)}\\
&+\delta_{10}(i,j)\frac{\mathbf{c}_1-\mathbf{x}_j}{d(\mathbf{c}_1,\mathbf{x}_j)}+\delta_{12}(i,j)\alpha(1,2)\frac{\mathbf{c}_1-\mathbf{c}_2}{d(\mathbf{c}_1,\mathbf{c}_2)}\Bigg)\\
&+p_2^2(i,j)\Bigg(\delta_{21}(i,j)\Bigg[-\alpha(1,2)\frac{\mathbf{c}_2-\mathbf{c}_1}{d(\mathbf{c}_2,\mathbf{c}_1)}+\frac{\mathbf{c}_1-\mathbf{x}_j}{d(\mathbf{c}_1,\mathbf{x}_j)}\Bigg]\Bigg)\Bigg]\\
= &\sum_{\substack{\{i,j\}\in\overline{1,N}\\ i\neq j}} w_{ij}\Bigg[\frac{\mathbf{c}_1-\mathbf{x}_i}{d(\mathbf{x}_i,\mathbf{c}_1)}p_1^2(i,j)\\
&+\frac{\mathbf{c}_1-\mathbf{x}_j}{d(\mathbf{c}_1,\mathbf{x}_j)}\big(p_1^2(i,j)\delta_{10}(i,j)+p_2^2(i,j)\delta_{21}(i,j)\big)\\
&+\alpha(1,2)\frac{\mathbf{c}_1-\mathbf{c}_2}{d(\mathbf{c}_1,\mathbf{c}_2)}\big(p_1^2(i,j)\delta_{12}(i,j)+p_2^2(i,j)\delta_{21}(i,j)\big)\Bigg].
\end{aligned} \tag{36}$$

Setting the gradient equal to zero, and summing like terms, we get

$$\begin{aligned}
&\Bigg(\sum_{\substack{\{i,j\}\in\overline{1,N}\\ i\neq j}} w_{ij}\Bigg[\frac{p_1^2(i,j)}{d(\mathbf{x}_i,\mathbf{c}_1)}+\frac{(p_1^2(i,j)\delta_{10}(i,j)+p_2^2(i,j)\delta_{21}(i,j))}{d(\mathbf{c}_1,\mathbf{x}_j)}\\
&\quad+\frac{\alpha(1,2)(p_1^2(i,j)\delta_{12}(i,j)+p_2^2(i,j)\delta_{21}(i,j))}{d(\mathbf{c}_1,\mathbf{c}_2)}\Bigg]\Bigg)\mathbf{c}_1\\
&=\sum_{\substack{\{i,j\}\in\overline{1,N}\\ i\neq j}} w_{ij}\Bigg[\frac{\mathbf{x}_i}{d(\mathbf{x}_i,\mathbf{c}_1)}p_1^2(i,j)+\frac{\mathbf{x}_j}{d(\mathbf{c}_1,\mathbf{x}_j)}\big(p_1^2(i,j)\delta_{10}(i,j)+p_2^2(i,j)\delta_{21}(i,j)\big)\\
&\quad+\alpha(1,2)\frac{\mathbf{c}_2}{d(\mathbf{c}_1,\mathbf{c}_2)}\big(p_1^2(i,j)\delta_{12}(i,j)+p_2^2(i,j)\delta_{21}(i,j)\big)\Bigg]
\end{aligned}$$

proving (29)–(33).

Here, the calculation of center in (HP.K) is analogous to the Weiszfeld center as in Iyigun and Ben-Israel (2010), except each hub center is not only convex combination of data points $\mathbf{x}_i$ but also the other hub.

Derivation of formulas for $\mathbf{c}_2$ can be shown similarly.

8 A clustering method for the hub location problem

The above results are implemented in an algorithm for solving (HP.K). A schematic description, presented for simplicity for the case of 2 hub centers, follows.

Algorithm 1 A clustering method for multi-assignment hub location problem

Data: $\mathbf{X} = \{\mathbf{x}_i : i \in \overline{1, N}\}$ data points (locations of cities),
$\mathbf{W} = \{ w_{ij} : i,j \in 1,N \}$ weights (demands) between data points,

K = 2, the number of hubs,

$\alpha(k, l),\ k \quad \ell, k, \ell \in \overline{1,}$ =2 discount factors between hubs,

> 0 (stopping criterion)

Initialization: K = 2 arbitrary hub centers $\{\mathbf{c}_k : k \in \overline{1, 2}\}$,

Iteration:

Step 1 **compute** costs of travel $\{c_k(\mathbf{x}_i, \mathbf{x}_j) : k \in \overline{1, 2}\}$ for all $\mathbf{x}_i, \mathbf{x}_j \in \mathbf{X}$ (using (14))

Step 2 **compute** probabilities $\{p_k(\mathbf{x}_i, \mathbf{x}_j) : k \in 1, 2\}$ for all $\mathbf{x}_i, \mathbf{x}_j \in \mathbf{X}$ (using (18))

Step 3 **update** the hub centers $\{ \mathbf{c}^+_k : k \in 1,2\}$ (using (29)&(34))

Step 4 **if** $\sum_{k=1}^{2} d(\mathbf{c}_k^+, \mathbf{c}_k) < \epsilon$ **stop**

return to step 1

9 Numerical examples

In order to illustrate the proposed algorithm, the test problem, German Towns (Späth 1980) from the literature is used and the results are shown below. The initial locations of hub centers $\{\mathbf{c}_k : k \in \overline{1, K}\}$ are taken randomly in the convex hull of the set of data points.

Example 2 This example uses the data of German Towns, originally presented by Späth (1980). It is required to locate 4 hub centers to serve the 59 towns shown in Fig. 2(a).

Algorithm 1 was tried, starting random initial centers, and using different discount factors α = 0.00,0.10,0.25,0.50,0.75,0.90,0.95,1.00 and a tolerance = 0.001. Here α = 0 indicates full discount and α = 1 indicates for no discount. The discount factor $\alpha_{kl}, k, \ell \in \overline{1, 4}$ between the hubs are taken equal and the demand w_{ij} between each pair of data point is assumed as a unit flow for both directions. Cost of direct connection is infinity in all instances.

The discount applied between the hub connections affect the locations of the hub centers and their serving to the demand points. Figures 2–4 show graphically the effect of the different values of discount factor on the hub locations and on the points that they serve for.

Since travelling between hubs is more economical with the lower values of α, the hub centers are located far away from each other and they get closer to the demand points. For $\alpha = 0$, the problem becomes a clustering problem or a multi-facility location problem, which is a particular case of HLP, each point is served only by the closest hub center. As the discount decreases (means higher α values), there is less advantage to travel between hubs and the hub locations get closer. Finally, when $\alpha = 1$ (no discount between hubs), HLP problem becomes a Fermat-Weber location problem and all hubs coincide, see Corollary 1.

The discount factor between hubs leads to the allocation of a demand point to multiple hubs. The number of data points served by multiple hubs increases with the higher values of α. As seen in Figs. 3(b), (c) and Figs. 4(a), (b), there is an expressive increase in the number of connections between cities and hubs when the value of α increases (that means travelling between the hubs is fully advantageous).

Example 3 We solved Example 2 for K = 2,3,4 using two discount values $\alpha = 0.2$ and 0.7. Again the flow demand between any two cities w_{ij} is 1. The solutions are shown in

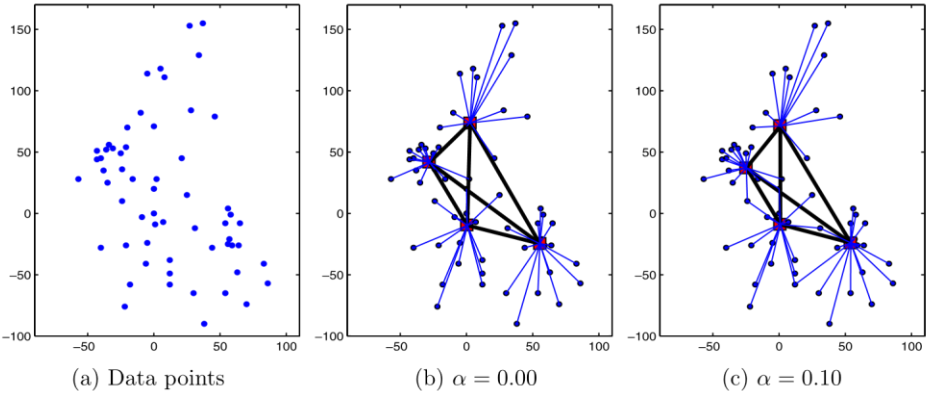
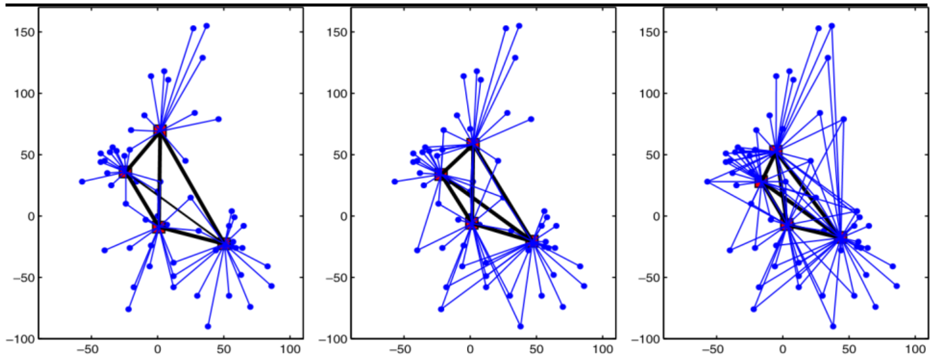
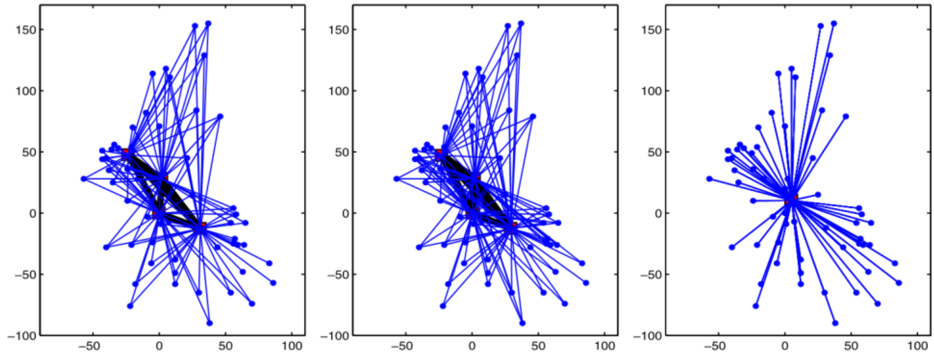


Fig. 2 Data points of Example 2 and solutions for cases K = 4 and $\alpha = 0.00, 0.10$



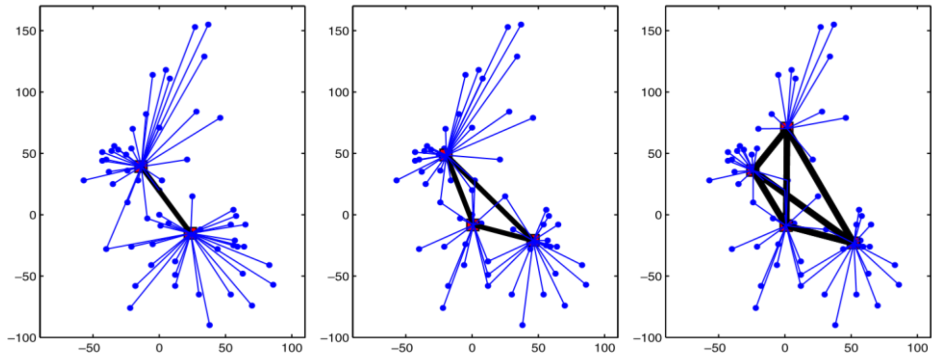
(a) $\alpha = 0.25$ (b) $\alpha = 0.50$ (c) $\alpha = 0.75$

Fig. 3 Solutions of Example 2 for cases K = 4 and α = 0.25,0.50,0.75



(a) $\alpha = 0.90$ (b) $\alpha = 0.95$ (c) $\alpha = 1.00$

Fig. 4 Solutions of Example 2 for cases K = 4 and α = 0.90,0.95,1.00



(a) $K = 2$ (b) $K = 3$ (c) $K = 4$

Fig. 5 Hub locations for Example 2 for α = 0.2 and K = 2,3,4

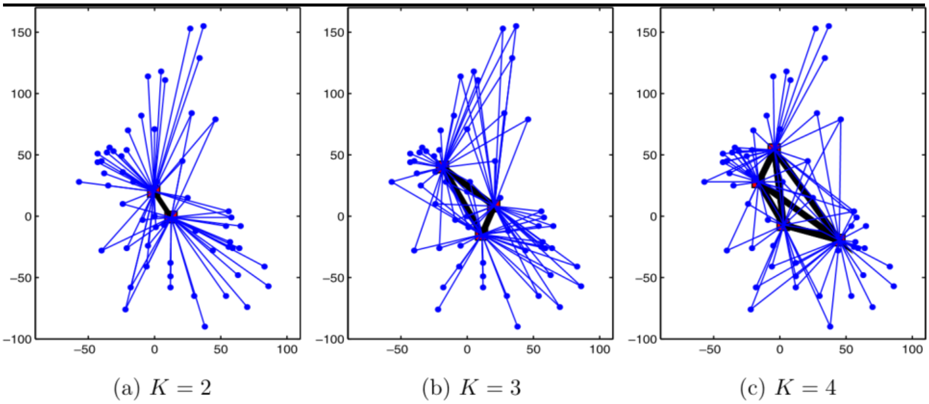


Fig. 6 Hub locations for Example 2 for α = 0.7 and K = 2,3,4

Fig. 5 and Fig. 6. For all cases of K, again hub locations are far away from each other in low values of α (high savings in travelling between hubs) and their locations are getting closer with high α value.

10 Conclusion

The planar hub location problem with multiple assignments has a non-convex and nondifferentiable objective function. The probabilistic clustering algorithm presented here proposes an approach for solving such a difficult problem. The algorithm is simple and requires a small number of cheap iterations. The numerical experiments show that the method can solve large instances of the problem in reasonable time.

The algorithm stems from a probabilistic clustering algorithm and it has a dependency on the initial locations of the hubs, which affects the quality of the final solution.

Since the planar hub location problem has not been studied extensively, as of our knowledge, there is no benchmark dataset in the literature. Results of our numerical experiments on simulated datasets and comparisons with other algorithms will be reported elsewhere.