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Extension of the Ginzburg–Landau approach for ultracold Fermi gases below a critical temperature ☆

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ABSTRACT

In the context of superfluid Fermi gases, the Ginzburg–Landau (GL) formalism for the macroscopic wave function has been successfully extended to the whole temperature range where the superfluid state exists. After reviewing the formalism, we first investigate the temperature-dependent correction to the standard GL expansion (which is valid close to T_c). Deviations from the standard GL formalism are particularly important for the kinetic energy contribution to the GL energy functional, which in turn influences the healing length of the macroscopic wave function. We apply the formalism to variationally describe vortices in a strong-coupling Fermi gas in the BEC–BCS crossover regime, in a two-band system. The healing lengths, derived as variational parameters in the vortex wave function, are shown to exhibit hidden criticality well below T_c.

1. Introduction

The Ginzburg–Landau (GL) approach is a powerful tool for the description of superconductors in the close vicinity of the critical temperature T_c. Recently, the GL method was re-derived in the context of superfluid ultracold Fermi gases [1–4]. The GL approach was also applied to explain the phenomenon of the "1.5-type" superconductivity. However, the validity of the GL approximation far below T_c is still under discussion [5]. In this connection, much efforts were undertaken to extend the GL approach to a wide range of temperatures (see, e.g., Refs. [6–11]).

In Ref. [12], we formulated an extension of the GL theory for a two-band superfluid fermion system solvable for the whole range $0 < T < T_c$ assuming slow variation of the order parameter in time and space, without any assumption on the magnitude of the order parameter. The theory is mainly focused to the strong-coupling ultracold atomic Fermi gases in the BCS–BEC crossover. In the present work, we briefly review the method and the description of vortices in a two-band system at temperatures $T \lesssim T_c$, where the standard GL technique is apparently inapplicable.

The formalism developed in Ref. [12] is aimed mainly at the investigation of localized deviations of the order parameters Ψ_j from a uniform equilibrium background Δ_j. These deviations can be, for example, vortices or solitons [13–19]. A frequently used theoretical method to study these localized deviations at temperatures far below T_c is a Bogoliubov–deGennes (BdG) equation set. Re-formulations of the BdG method for ultracold atoms can be found, e.g., in Refs. [20–23]. The present method can be used as a complementary tool to the BdG equations and is straightforward to implement numerically. Moreover, the BdG equations are restricted to the mean-field approach, while the present method can be used beyond the mean-field approximation accounting for fluctuations about the saddle point.

2. Formalism

The starting point of our treatment is the partition function of a two-band fermion system in the path-integral representation,

$$Z \propto \int D[\bar{\psi},\psi]\, e^{-S}, \tag{1}$$

with the fermion action depending on the Grassmann fields $\bar{\psi}$, ψ,

$$S = S_0 + \int_0^\beta d\tau \int d\mathbf{r} U(\mathbf{r},\tau). \tag{2}$$

Here, S_0 is the free-fermion action functional,

$$S_0 = \int_0^\beta d\tau \int d\mathbf{r} \sum_{j,\sigma=\uparrow,\downarrow} \bar{\psi}_{\sigma j}\left(\frac{\partial}{\partial\tau} - \frac{\nabla_{\mathbf{r}}^2}{2m_j} - \mu_{\sigma j}\right)\psi_{\sigma j}, \tag{3}$$

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accounting for both spin and band imbalance of the fermion system – through unequal masses m_j and chemical potentials μ_j. The fermion–fermion interaction U is given by:

$$U = \sum_{j=1,2} g_j \bar{\psi}_{\uparrow,j}\bar{\psi}_{\downarrow,j}\psi_{\downarrow,j}\psi_{\uparrow,j} + g_3\left(\bar{\psi}_{\uparrow,1}\psi_{\uparrow,1}\bar{\psi}_{\downarrow,2}\psi_{\downarrow,2} + \bar{\psi}_{\downarrow,1}\psi_{\downarrow,1}\bar{\psi}_{\uparrow,2}\psi_{\uparrow,2}\right) + g_4\left(\bar{\psi}_{\uparrow,1}\psi_{\uparrow,1}\bar{\psi}_{\uparrow,2}\psi_{\uparrow,2} + \bar{\psi}_{\downarrow,1}\psi_{\downarrow,1}\bar{\psi}_{\downarrow,2}\psi_{\downarrow,2}\right). \quad (4)$$

It contains the terms describing both the intraband s-wave scattering (with $j = 1,\ 2$) and the scattering between fermions in different bands (with $j = 3,\ 4$). Introducing the auxiliary bosonic fields and performing the Hubbard–Stratonovich transformation we arrive at an effective bosonic action of the pair fields as described in Ref. [12]. Subsequently, we make the standard approximation for the GL approach: we assume that the pair fields slowly vary in time and space. The gradient expansion of the pair fields leads to the long-wavelength approximation for the effective bosonic action. This yields the following GL-like free energy

$$F = \frac{1}{\beta}\int_0^\beta d\tau \int d\mathbf{r}\left\{\sum_{j=1,2}\left[\Omega_{s,j} + \frac{\mathcal{D}_j}{2}\left(\bar{\Psi}_j\frac{\partial\Psi_j}{\partial\tau} - \frac{\partial\bar{\Psi}_j}{\partial\tau}\Psi_j\right) + \frac{\mathcal{C}_j}{2m_j}\left|\nabla_{\mathbf{r}}\Psi_j\right|^2 - \frac{\mathcal{E}_j}{2m_j\left|\Psi_j\right|^2}\left[\left(\bar{\Psi}_j\nabla_{\mathbf{r}}\Psi_j\right)^2 + \left(\Psi_j\nabla_{\mathbf{r}}\bar{\Psi}_j\right)^2\right]\right] - \frac{\sqrt{m_1 m_2}\gamma}{4\pi}\left(\bar{\Psi}_1\Psi_2 + \bar{\Psi}_2\Psi_1\right)\right\}. \quad (5)$$

Here, the function $\Omega_{s,j}$ formally coincides with the saddle-point thermodynamic potential for the imbalanced Fermi gas,

$$\Omega_{s,j} = -\int\frac{d\mathbf{k}}{(2\pi)^3}\left[\frac{1}{\beta}\ln\left(2\cosh\beta E_{\mathbf{k},j} + 2\cosh\beta\zeta_j\right) - \xi_{\mathbf{k},j} - \frac{m_j\left|\Psi_j\right|^2}{k^2}\right] - \frac{m_j\left|\Psi_j\right|^2}{4\pi a_j}, \quad (6)$$

where $E_{\mathbf{k},j} = \sqrt{\xi_{\mathbf{k},j}^2 + \left|\Psi_j\right|^2}$ is the Bogoliubov excitation energy, $\xi_{\mathbf{k},j} = \frac{k^2}{2m_j} - \mu_j$ is the free-fermion energy, and the chemical potentials for the imbalanced fermions are $\mu_j = \left(\mu_{j,\uparrow} + \mu_{j,\downarrow}\right)/2$ and $\zeta_j = \left(\mu_{j,\uparrow} - \mu_{j,\downarrow}\right)/2$. However, the order parameter Ψ_j entering this thermodynamic potential is coordinate-dependent. The parameter γ describes the strength of coupling between two bands. The coefficients $\mathcal{C}_j$, $\mathcal{D}_j$ and $\mathcal{E}_j$ derived in Ref. [12] are

$$\mathcal{C}_j = \int d\mathbf{k}\frac{k^2\left[f_2(\beta, E_{\mathbf{k},j}, \zeta_j) - 4\xi_k^2\left|\Psi_j\right|^2 f_4(\beta, E_{\mathbf{k},j}, \zeta_j)\right]}{24\pi^3 m_j}, \quad (7)$$

$$\mathcal{D}_j = \int\frac{d\mathbf{k}}{(2\pi)^3}\frac{\xi_{\mathbf{k}}}{\left|\Psi_j\right|^2}\left[f_1(\beta, \xi_{\mathbf{k},j}, \zeta_j) - f_1(\beta, E_{\mathbf{k},j}, \zeta_j)\right], \quad (8)$$

$$\mathcal{E}_j = 2\left|\Psi_j\right|^2\int\frac{d\mathbf{k}}{(2\pi)^3}\frac{k^2}{3m_j}\xi_{\mathbf{k},j}^2 f_4(\beta, E_{\mathbf{k},j}, \zeta_j). \quad (9)$$

The functions $f_p(\beta, \varepsilon, \zeta)$ are the Matsubara sums:

$$f_p(\beta, \varepsilon, \zeta) = \sum_{n=-\infty}^{\infty}\frac{1}{\left(\left(\omega_n - i\zeta\right)^2 + \varepsilon^2\right)^p}. \quad (10)$$

They can be analytically expressed, e.g., using the recurrence relations:

$$f_1(\beta, \varepsilon, \zeta) = \frac{1}{2\varepsilon}\frac{\sinh\beta\varepsilon}{\cosh\beta\varepsilon + \cosh\beta\zeta}, \quad (11)$$

$$f_{p+1}(\beta, \varepsilon, \zeta) = -\frac{1}{2p\varepsilon}\frac{\partial f_p(\beta, \varepsilon, \zeta)}{\partial\varepsilon}. \quad (12)$$

In order to analytically compare the results of the present approach with the known GL method near T_c, we use the results of Ref. [24], which represents the limiting case of the present approach when $T \to T_c$ (for a one-band system and without imbalance). For temperatures near T_c, the order parameter is small. Thus the coordinate-dependent thermodynamic potential $\Omega_{s,j}$ is expanded in powers of $\left|\Psi_j\right|^2$ up to the quartic order, and the coefficients $\mathcal{C}_j$, $\mathcal{D}_j$ and $\mathcal{E}_j$ are kept for $\Psi_j = 0$. In this case, the GL-like free energy (5) is reduced to the TDGL free energy of Ref. [24], except for the coefficient $\mathcal{D}$, which appears to be real in the present approach. The reason for this difference consists in the following. The imaginary part in $\mathcal{D}$ appears in Ref. [24] when the gradient expansion is performed at $\Psi = 0$. On the contrary we perform the summations of the whole series in powers of the order parameter *before* taking the limit $T \to T_c$, indicating therefore that the appearance of an imaginary part of $\mathcal{D}$ depends on a sequence of the limits $\Psi \to 0$ and $T \to T_c$.

3. Results

First, we illustrate a difference of the temperature behavior of the coefficients of the GL-like free energy (5) compared to the coefficients of the TDGL equation of Ref. [24].

In Fig. 1, the coefficient $\mathcal{C}$ is plotted as a function of temperature for several values of the inverse scattering length $1/a_s$ and compared with the coefficient c of Ref. [24]. Both coefficients analytically tend to the same values when $T \to T_c$. The temperature behavior of the coefficient $\mathcal{C}$ drastically differs from that for the corresponding coefficient c within the GL approach [24]. In the whole range of the BCS–BEC crossover, the coefficient $\mathcal{C}$ only slightly varies when T goes from T_c to zero. When increasing the inverse scattering length, the range of temperatures, where $\mathcal{C}$ and c are rather close to each other, gradually broadens. In the molecular (BEC) regime, both solutions tend to one and the same limit for all $T \leqslant T_c$. On the contrary, for $1/a_s \leq 0$, (i, e., at the BCS side and at unitarity) c rapidly increase when decreasing temperature (except for the BEC case), and even diverges at $T \to 0$. These results confirm the fact that the standard GL approach becomes inapplicable at low temperatures.

Vortices are studied in the present work using the variational method. The deviations of the order parameters Ψ_j from a uniform equilibrium background can be represented through the product of

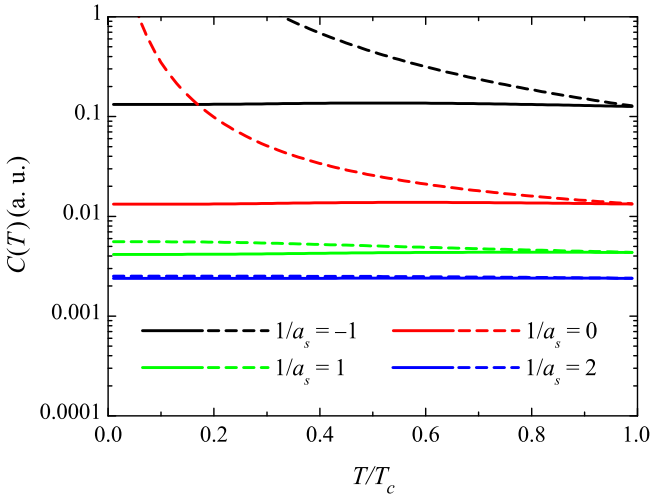

Fig. 1. The coefficient C calculated, as a function of temperature, within the extended TDGL formalism (solid curves) and within the TDGL theory of Ref. [24] (dashed curves).

a uniform background amplitude $\Delta_j \equiv \left|\Psi_j^{bulk}\right|$ with the amplitude modulation function $f_j(\mathbf{r},\tau)$ and the phase factor $e^{i\theta_j(\mathbf{r},\tau)}$:

$$\Psi_j = \Delta_j \cdot f_j(\mathbf{r},\tau) e^{i\theta_j(\mathbf{r},\tau)}. \tag{13}$$

The coefficients $\mathcal{D}_j$, $\mathcal{C}_j$, $\mathcal{E}_j$ are kept with the bulk values of the order parameter. Thus the time and space dependence are taken in leading order through the derivatives. This is in line with the gradient-expansion approximation which was already kept when deriving (5).

Further on, we introduce the notations:

$$\rho_j^{(qp)} = \frac{(\mathcal{C}_j - 2\mathcal{E}_j)|\Delta_j|^2}{m_j}, \tag{14}$$

$$\rho_j^{(sf)} = \frac{(\mathcal{C}_j + 2\mathcal{E}_j)|\Delta_j|^2}{m_j}. \tag{15}$$

The parameter $\rho_j^{(sf)}$ is the superfluid density, and $\rho_j^{(qp)}$ is the quantum pressure coefficient.

Using (13), we arrive at the following variational GL-like free energy functional,

$$F = \frac{1}{\beta}\int_0^\beta d\tau \int d\mathbf{r}\left\{\sum_{j=1,2}\left[\Omega_{s,j}(w_j) + i\mathcal{D}_j|\Delta_j|^2 f_j^2\frac{\partial\theta_j}{\partial\tau} + \frac{1}{2}\rho_j^{(qp)}(\nabla f_j)^2 + \frac{1}{2}\rho_j^{(sf)} f_j^2(\nabla\theta_j)^2\right] - \frac{\sqrt{m_1 m_2}\gamma}{2\pi}\Delta_1\Delta_2 f_1 f_2\cos(\theta_2-\theta_1)\right\}. \tag{16}$$

It describes, in principle, not only the stationary states but also the time-dependent Josephson physics for a two-band system due to the phase difference $\theta_2(\mathbf{r},\tau) - \theta_1(\mathbf{r},\tau)$.

In the present work, the amplitude modulation function for a vortex was used in the form

$$f(r,\xi) = \tanh\left(\frac{r}{\sqrt{2}\xi}\right), \tag{17}$$

with the healing length ξ. The healing lengths are determined minimizing the free energy (16). In Fig. 2a, the healing lengths for a vortex in a two-band system are plotted as a function of temperature for the inverse scattering lengths (in units of the Fermi wave vector k_F) $1/a_1 = 0$, $1/a_2 = -0.5$, and for different values of the coupling parameter γ. The healing length for the "stronger" band, ξ_1, extremely weakly depends on γ. The healing length for the "weaker" band, ξ_2, demonstrates the "hidden criticality" discussed in our manuscript and in Ref. [25]. At zero interband coupling, each of two subsystems (the "stronger" and "weaker" bands) is characterized by its own critical temperature $T_{c,j}$ and healing length ξ_j, which tends to infinity at $T \to T_{c,j}$. When the Josephson interband coupling is nonzero but sufficiently weak, we can see a fingerprint of the phase transition for a "weaker" band as a peak of the healing length ξ_2 at $T \approx T_{c,2}$.

When comparing the healing lengths for a vortex calculated in the present work note with those calculated in Ref. [12] using the model fermion system near a hard wall, we see a qualitative agreement between the healing lengths determined by these two methods. However, there is some quantitative difference between these healing lengths.

In Fig. 2b, the ratio ξ_2/ξ_1 is plotted for the same parameters as in Fig. 1. We can note on a remarkable similarity between these results and those shown in Fig. 2c of Ref. [25] for a two-band superconductor using BdG equations. The ratio ξ_2/ξ_1 starts from a value $\xi_2/\xi_1 > 1$ at zero temperature, exhibits a peak near the critical temperature for a "weaker" band $T_{c,2}$, and tends to 1 when $T \to T_c$ (which is very close to $T_{c,1}$).

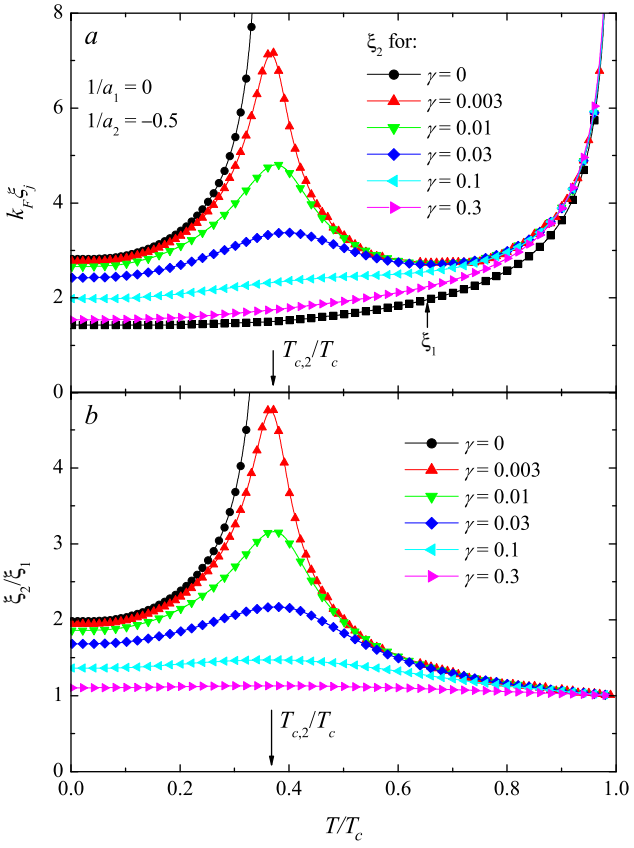

Fig. 2. (a) Healing lengths for a two-band superfluid fermion system as a function of temperature for different values of the coupling parameter γ and (b) the ratio of the healing lengths ξ_2/ξ_1.

4. Conclusions

In summary, we re-formulated the path-integral approach for interacting Fermi gases [24] to the case of a two-band system. The Hubbard–Stratonovich transformation and the integration over the fermion fields lead to an effective bosonic action with Josephson interband coupling. The gradient expansion of the effective bosonic action results in the GL-like free energy functional in which the amplitude of the pair field is not a small parameter. Therefore the obtained free energy represents an extension of the Ginzburg–Landau formalism to temperatures below T_c. The range of applicability of the gradient expansion is determined by the same conditions as for the standard GL approach, where that expansion is also used. Thus the present extended GL-like method is valid under the same conditions as the GL approach – but in a wider temperature range.

As an example, the method has been tested for vortices in a two-band system of ultracold fermions. It has been shown that the "hidden criticality" far below T_c, treated previously using the BdG equations [25] is captured by the extended GL-like approach. Because of the validity of the present approach at temperatures far below T_c, it can find a wide spectrum of applications, e.g., for the analysis of distributions of trapped fermionic atoms, vortices, solitons and other spatially non-uniform phenomena in ultracold Fermi gases.

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