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STEP-BY-STEP SIMPLIFICATION OF THE MICROPOLAR ELASTICITY THEORY TO THE COUPLE-STRESS AND CLASSICAL ELASTICITY THEORIES

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ABSTRACT

The micropolar elasticity theory provides a useful material model for dealing with fibrous, coarse granular, and large molecule materials. Though being a well-known and well-developed elasticity model, the linear theory of micropolar elasticity is not without controversy. Specially simplification of the microppolar elasticity theory to the couple-stress and classical elasticity theories and the required conditions on the material elastic constants for this simplification have not been discussed consistently. In this paper the linear theory of micropolar elasticity is reviewed first. Then the correct approach for a consistent and step-by-step simplification of the micropolar elasticity model with six elastic constants to the couple-stress elasticity model with four elastic constants and the classical elasticity model with two elastic constants is presented. It is shown that the classical elasticity is a special case of the couple-stress theory which itself is a special case of the micropolar elasticity theory.

INTRODUCTION

The classical theory of linear elasticity has a long history of development and verification and produces acceptable results in numerous engineering problems with various structural materials. However, for the cases with large stress gradients (e.g., in the vicinity of holes and cracks) or materials with significant microstructure contribution (e.g. composites, polymers, soil, and bone) the classical theory of elasticity fails to produce acceptable results. To improve the results of the classical theory of elasticity Voit [1] incorporated the effects of couple stresses and generalized the symmetric classical theory of elasticity to the asymmetric couple-stress theory. This was then extended by E. and F. Cosserat [2] who considered a body microrotation field, independent of the body displacement field. Eringen [3] further developed Cosserat's model by including the body microinertia and renamed it as the micropolar theory of (asymmetric) elasticity. Nowacki [4] provides an extensive description of the linear theory of micropolar elasticity.

Broadly speaking these newer, more elaborate, material models are useful when dealing with materials that have a defined internal structure; *e.g.* fibrous materials such as bone, coarse granular materials such as soil, and large molecule polymers such as foams. It is noteworthy that experimental verification of the micropolar theory for these materials is not fully accomplished yet and one is faced with a situation when theory precedes experiment.

Although being a well-known and well-developed elasticity model, the linear theory of micropolar elasticity is not without controversy. Especially simplification of the micropolar elasticity to the classical elasticity theory and determination of the micropolar material parameters are labeled as inconsistent. In particular, whereas the micropolar elasticity model with zero micropolar elastic constants (including a zero micropolar couple modulus) is considered by some authors to coincide with the classical elasticity model (*e.g.* [5], [6], and [7]), there are other authors who observed some inconsistencies in the micropolar elasticity model with a zero micropolar couple modulus (*e.g.* [8] and [7, 9]).

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This paper suggests that the apparent inconsistencies in the micropolar theory of linear elasticity are mainly a result of the approach taken or assumed for simplification of the micropolar theory of elasticity to the classical theory of elasticity and may be resolved provided a different simplification approach is taken. We will show, assuming a general linear theory of micropolar elasticity applied to a homogeneous, isotropic, and centrally symmetric material, that a new approach, in which a zero micropolar couple modulus is not required, can be taken for simplification of the micropolar theory of elasticity to the classical theory of elasticity.

MICROPOLAR ELASTICITY THEORY

To provide a brief overview of the three-dimensional linear theory of micropolar elasticity, consider a general homogeneous, isotropic, and centrally symmetric elastic body occupying a volume domain V in $\mathbb{R}^3$, bounded by surface S. Assume that the body undergoes a motion and deformation due to the action of external volume force and moment $\underset{\rightarrow}{f}^V$ and $\underset{\rightarrow}{m}^V$. A body frame $\mathscr{F}_b$ and a position vector $\underset{\rightarrow}{p}$ (with respect to the inertial frame $\mathscr{F}_o$) correspond to each representative infinitesimal element of the body (see Fig. 1).

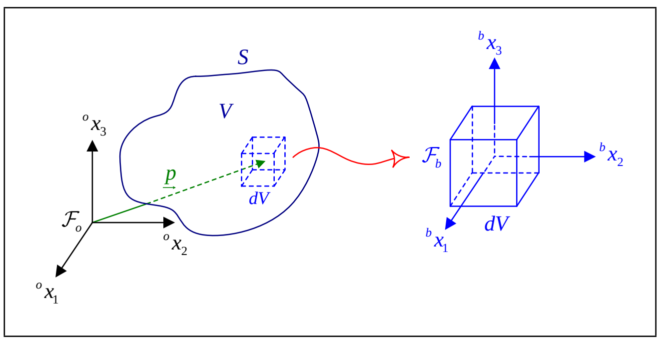

FIGURE 1. A GENERAL ELASTIC BODY AND ITS REPRESENTATIVE ELEMENT.

In the micropolar elasticity model, the (classical) displacement field vector $\underset{\rightarrow}{u}$ is complemented by a microrotation field vector $\underset{\rightarrow}{\vartheta}$ (independent of the displacement field). Consequently, the translational velocity and acceleration vectors are $\underset{\rightarrow}{\dot{u}}$ and $\underset{\rightarrow}{\ddot{u}}$, and the angular velocity and acceleration vectors are $\underset{\rightarrow}{\dot{\vartheta}}$ and $\underset{\rightarrow}{\ddot{\vartheta}}$ [10].

The micropolar deformation is fully described by (asymmetric) strain and twist tensors, $\underset{\leftrightarrow}{\varepsilon}$ and $\underset{\leftrightarrow}{\tau}$, which are defined as:

$$\begin{aligned} \varepsilon_{ij} &= u_{j,i} - e_{ijk}\,\vartheta_k, \\ \tau_{ij} &= \vartheta_{j,i}, \end{aligned} \tag{1}$$

where e_{ijk} is the third-order Levi-Civita or permutation tensor. Based on these definitions the following relations can be derived:

$$\begin{aligned} \vartheta_i &= \frac{1}{2}\,e_{ijk}\left(u_{k,j} - \varepsilon_{jk}\right), \\ \tau_{ij} &= \frac{1}{2}\,e_{jkl}\left(u_{l,ki} - \varepsilon_{kl,i}\right), \\ \tau_{ii} &= -\frac{1}{2}\,e_{ijk}\,\varepsilon_{jk,i}. \end{aligned} \tag{2}$$

It is also useful to define the (classical) macrorotation vector $\underset{\rightarrow}{\theta}$ and the (classical) macrorotation tensor $\underset{\leftrightarrow}{\theta}^\times$ such that:

$$\begin{aligned} \theta_i &= \frac{1}{2} e_{ijk} u_{k,j}, \\ \theta^\times_{ij} &= -\,e_{ijk}\,\theta_k = -\frac{1}{2}\left(u_{j,i} - u_{i,j}\right). \end{aligned} \tag{3}$$

Then the strain and twist tensors can be decomposed into their symmetric and antisymmetric (skew-symmetric) parts as:

$$\begin{aligned} \varepsilon_{ij} &= \varepsilon^{\mathrm{s}}_{ij} + \varepsilon^{\mathrm{a}}_{ij}, \\ \varepsilon^{\mathrm{s}}_{ij} &= \frac{1}{2}\left(u_{j,i} + u_{i,j}\right), \\ \varepsilon^{\mathrm{a}}_{ij} &= \frac{1}{2}\left(u_{j,i} - u_{i,j}\right) - e_{ijk}\,\vartheta_k = e_{ijk}\left(\theta_k - \vartheta_k\right), \end{aligned} \tag{4}$$

and:

$$\begin{aligned} \tau_{ij} &= \tau^{\mathrm{s}}_{ij} + \tau^{\mathrm{a}}_{ij}, \\ \tau^{\mathrm{s}}_{ij} &= \frac{1}{2}\left(\vartheta_{j,i} + \vartheta_{i,j}\right), \\ \tau^{\mathrm{a}}_{ij} &= \frac{1}{2}\left(\vartheta_{j,i} - \vartheta_{i,j}\right), \end{aligned} \tag{5}$$

where note that $\underset{\leftrightarrow}{\varepsilon}^{\mathrm{a}}$ is a representation of the difference between the (classical) macrorotation and the (micropolar) microrotation.

As shown in Fig. 2, in a micropolar continuum, the (classical force) stress field tensor $\underset{\leftrightarrow}{\sigma}$ is completed by a (micropolar) couple stress field tensor $\underset{\leftrightarrow}{\chi}$. Considering the free body diagram in Fig. 2, for a micropolar elastic body under the action of a general volume force $\underset{\rightarrow}{f}^V$ and a general volume moment $\underset{\rightarrow}{m}^V$ the balance of linear and angular momenta can be written in the following differential form:

$$\begin{aligned} \sigma_{ji,j} + f^V_i &= \rho^V \ddot{u}_i, \\ \chi_{ji,j} + e_{ijk}\,\sigma_{jk} + m^V_i &= \iota^V \ddot{\vartheta}_i, \end{aligned} \tag{6}$$

where ρ^V is the material volume mass density and ι^V is the material volume microinertia density. Note that a more general case is when the material has a tensor of microinertia density $\underset{\leftrightarrow}{\iota}^V$, however this paper is confined to the isotropic case where $\underset{\leftrightarrow}{\iota}^V = \iota^V \underset{\leftrightarrow}{\mathbb{1}}$ [4].

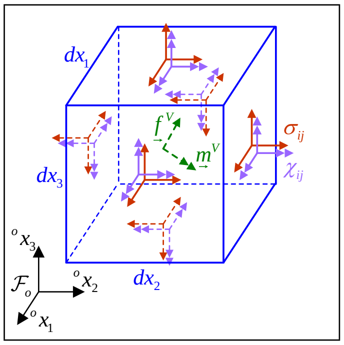

FIGURE 2. FREE BODY DIAGRAM OF A REPRESENTATIVE ELEMENT IN A MICROPOLAR ELASTIC BODY.

Analogous to the decomposition of strain and twist tensors, the force and couple stress tensors can be decomposed into their symmetric and antisymmetric parts as:

$$\begin{aligned} &\sigma_{ij} = \sigma^{\mathrm{s}}_{ij} + \sigma^{\mathrm{a}}_{ij}, \\ &\sigma^{\mathrm{s}}_{ij} = \frac{1}{2}\Big(\sigma_{ij} + \sigma_{ji}\Big), \qquad \sigma^{\mathrm{a}}_{ij} = \frac{1}{2}\Big(\sigma_{ij} - \sigma_{ji}\Big), \end{aligned} \tag{7}$$

and:

$$\begin{aligned} &\chi_{ij} = \chi^{\mathrm{s}}_{ij} + \chi^{\mathrm{a}}_{ij}, \\ &\chi^{\mathrm{s}}_{ij} = \frac{1}{2}\Big(\chi_{ij} + \chi_{ji}\Big), \qquad \chi^{\mathrm{a}}_{ij} = \frac{1}{2}\Big(\chi_{ij} - \chi_{ji}\Big). \end{aligned} \tag{8}$$

Utilizing Eqn. (7) and noting the fact that for any symmetric second-order tensor $\underset{\leftrightarrow}{d}$, $e_{ijk} d_{jk} = 0$, the equilibrium relations in Eqn. (6) can be rewritten as:

$$\begin{aligned} \sigma^{\mathrm{s}}_{ji,j} + \sigma^{\mathrm{a}}_{ji,j} + f^V_i = \rho^V \ddot{u}_i, \\ \chi_{ji,j} + e_{ijk}\,\sigma^{\mathrm{a}}_{jk} + m^V_i = \iota^V \ddot{\vartheta}_i. \end{aligned} \tag{9}$$

Solving the second relation of Eqn. (9) for the antisymmetric force stress tensor $\underset{\leftrightarrow}{\sigma}^{\mathrm{a}}$ and substituting into the first relation of Eqn. (9) one can rewrite the balance relations as:

$$\begin{aligned} \sigma^{\mathrm{s}}_{ji,j} + \frac{1}{2} e_{ijk}\Big(\chi_{lk,lj} + m^V_{k}{}_{,j} - \iota^V \ddot{\vartheta}_{k,j}\Big) + f^V_i = \rho^V \ddot{u}_i, \\ e_{ijk}\Big(\chi_{lk,l} + m^V_k - \iota^V \ddot{\vartheta}_k\Big) = 2\,\sigma^{\mathrm{a}}_{ji}. \end{aligned} \tag{10}$$

As can be concluded from Eqn. (9) the antisymmetric part of the force stress tensor, $\underset{\leftrightarrow}{\sigma}^{\mathrm{a}}$, couples the linear and angular momenta balance relations.

The linear theory of micropolar elasticity proposed by Eringen results in a set of two constitutive relations with six elastic constants for a general homogeneous, isotropic, and centrally symmetric elastic body. These relations have the following form:

$$\begin{aligned} \sigma_{ij} &= \Big(\mu + \kappa\Big)\,\varepsilon_{ij} + \Big(\mu - \kappa\Big)\,\varepsilon_{ji} + \lambda\,\varepsilon_{kk}\,\mathbb{1}_{ij}, \\ \chi_{ij} &= \Big(\gamma + \beta\Big)\,\tau_{ij} + \Big(\gamma - \beta\Big)\,\tau_{ji} + \alpha\,\tau_{kk}\,\mathbb{1}_{ij}, \end{aligned} \tag{11}$$

where $\mathbb{1}_{ij}$ is the Kronecker delta tensor. Among the six elastic constants denoted in Eqn. (11), μ and λ are the classical Lamé parameters (μ is also called Lamé shear modulus or shear modulus). The other four constants κ, γ, β, and α are the new elastic constants usually referred to as the micropolar or Cosserat elastic constants. Note that κ is usually called the micropolar couple modulus. The micropolar constants represent the contribution of the material microstructure to the elastic properties of the body.

By decomposing the strain and twist tensors, as given by Eqns. (4) and (5), the constitutive relations in Eqn. (11) can be rewritten as:

$$\begin{aligned} \sigma_{ij} &= 2\,\mu\,\varepsilon^{\mathrm{s}}_{ij} + \lambda\,\varepsilon_{kk}\,\mathbb{1}_{ij} + 2\,\kappa\,\varepsilon^{\mathrm{a}}_{ij}, \\ \chi_{ij} &= 2\,\gamma\,\tau^{\mathrm{s}}_{ij} + \alpha\,\tau_{kk}\,\mathbb{1}_{ij} + 2\,\beta\,\tau^{\mathrm{a}}_{ij}. \end{aligned} \tag{12}$$

Now a similar decomposition for the force and couple stress tensors, as given by Eqns. (7) and (8), gives rise to the following relations:

$$\begin{aligned} \sigma^{\mathrm{s}}_{ij} &= 2\,\mu\,\varepsilon^{\mathrm{s}}_{ij} + \lambda\,\varepsilon_{kk}\,\mathbb{1}_{ij}, \qquad \sigma^{\mathrm{a}}_{ij} = 2\,\kappa\,\varepsilon^{\mathrm{a}}_{ij}, \\ \chi^{\mathrm{s}}_{ij} &= 2\,\gamma\,\tau^{\mathrm{s}}_{ij} + \alpha\,\tau_{kk}\,\mathbb{1}_{ij}, \qquad \chi^{\mathrm{a}}_{ij} = 2\,\beta\,\tau^{\mathrm{a}}_{ij}, \end{aligned} \tag{13}$$

where the first relation is identical with the constitutive relation of classical elasticity.

Considering the original constitutive relations in Eqn. (11) and applying the Einstein summation convention on stress tensors, $\underset{\leftrightarrow}{\sigma}$ and $\underset{\leftrightarrow}{\chi}$, it can be shown that:

$$\begin{aligned} \sigma_{kk} &= 3\,B\,\varepsilon_{kk}, \qquad B = \lambda + \frac{2}{3}\,\mu, \\ \chi_{kk} &= 3\,\mathscr{B}\,\tau_{kk}, \qquad \mathscr{B} = \alpha + \frac{2}{3}\,\gamma, \end{aligned} \tag{14}$$

where B is known as the bulk modulus, and $\mathscr{B}$ as dual of the bulk modulus can be called the tortile or torsional bulk modulus.

Simplifying the first constitutive relation given by Eqn. (11) for the simple force stress state of uniform tension along axis ${}^{o}x_1$, where the only nonzero element of the force stress tensor is σ_{11}, results in definitions of the (classical) strain Poisson's ratio ν and the (classical) Young's modulus E as:

$$\begin{aligned} \nu &= -\frac{\varepsilon_{22}}{\varepsilon_{11}} = -\frac{\varepsilon_{33}}{\varepsilon_{11}} = \frac{\lambda}{2\left(\mu+\lambda\right)}, \\ E &= \frac{\sigma_{11}}{\varepsilon_{11}} = \frac{\mu\left(2\mu+3\lambda\right)}{\mu+\lambda} = 2\mu\left(1+\nu\right). \end{aligned} \tag{15}$$

The second constitutive relation in Eqn. (11) can be simplified to account for the simple couple stress state of uniform torsion along axis ${}^{o}x_1$, where the only nonzero element of the couple stress tensor is χ_{11}. Then the micropolar twist Poisson's ratio ξ and the micropolar tortile or torsional modulus $\mathscr{E}$ can be defined as:

$$\begin{aligned} \xi &= -\frac{\tau_{22}}{\tau_{11}} = -\frac{\tau_{33}}{\tau_{11}} = \frac{\alpha}{2\left(\gamma+\alpha\right)}, \\ \mathscr{E} &= \frac{\chi_{11}}{\tau_{11}} = \frac{\gamma\left(2\gamma+3\alpha\right)}{\gamma+\alpha} = 2\gamma\left(1+\xi\right). \end{aligned} \tag{16}$$

Utilizing the constitutive relations in Eqn. (11) and the definitions of the strain and twist tensors in Eqn. (1) to replace the force and couple stresses in the balance of momenta relations given by Eqn. (6), the system of partial differential equations (PDEs) representing the equations of motion for a micropolar continuum are derived as:

$$\begin{aligned} &\left(\mu+\kappa\right) u_{i,jj} + \left(\mu-\kappa+\lambda\right) u_{j,ji} \\ &\qquad + 2\kappa\, e_{ijk}\, \vartheta_{k,j} + f_i^V = \rho^V \ddot{u}_i, \\ &\left(\gamma+\beta\right) \vartheta_{i,jj} + \left(\gamma-\beta+\alpha\right) \vartheta_{j,ji} \\ &\qquad + 2\kappa\left(e_{ijk}\, u_{k,j} - 2\vartheta_i\right) + m_i^V = \iota^V \ddot{\vartheta}_i. \end{aligned} \tag{17}$$

Considering Eqns. (9) and (10) and substituting from the associated constitutive equations and the definitions of the strain and twist tensors into them, the following alternative forms for the dynamic equations can be attained respectively:

$$\begin{aligned} &\mu\, u_{i,jj} + \left(\mu+\lambda\right) u_{j,ji} - 2\kappa\, e_{ijk}\left(\theta_{k,j} - \vartheta_{k,j}\right) \\ &\qquad + f_i^V = \rho^V \ddot{u}_i, \\ &\left(\gamma+\beta\right) \vartheta_{i,jj} + \left(\gamma-\beta+\alpha\right) \vartheta_{j,ji} + 4\kappa\left(\theta_i - \vartheta_i\right) \\ &\qquad + m_i^V = \iota^V \ddot{\vartheta}_i, \end{aligned} \tag{18}$$

and:

$$\begin{aligned} &\mu\, u_{i,jj} + \left(\mu+\lambda\right) u_{j,ji} \\ &+ \frac{1}{2} e_{ijk}\left(\left(\gamma+\beta\right)\vartheta_{k,llj} + m_{k,j}^V - \iota^V \ddot{\vartheta}_{k,j}\right) + f_i^V = \rho^V \ddot{u}_i, \\ &\left(\gamma+\beta\right) \vartheta_{i,jj} + \left(\gamma-\beta+\alpha\right) \vartheta_{j,ji} \\ &\qquad + m_i^V - \iota^V \ddot{\vartheta}_i = -4\kappa\left(\theta_i - \vartheta_i\right), \end{aligned} \tag{19}$$

where it is recalled that for any symmetric second-order tensor $\underset{\leftrightarrow}{d}$, $e_{ijk} d_{jk} = 0$.

In the linear micropolar elasticity theory the strain energy density $\mathscr{U}_e{}^V$ is expressed as:

$$2\,\mathscr{U}_e{}^V = \sigma_{ij}\,\varepsilon_{ij} + \chi_{ij}\,\tau_{ij}. \tag{20}$$

By decomposing the force stress, couple stress, strain, and twist tensors into their symmetric and antisymmetric parts this expression can be rewritten as:

$$2\,\mathscr{U}_e{}^V = \sigma_{ij}^{s}\,\varepsilon_{ij}^{s} + \sigma_{ij}^{a}\,\varepsilon_{ij}^{a} + \chi_{ij}^{s}\,\tau_{ij}^{s} + \chi_{ij}^{a}\,\tau_{ij}^{a}. \tag{21}$$

Correspondingly substitutions from the constitutive relations in Eqns. (11) and (13) into Eqns. (20) and (21) result in the following strain energy density expressions:

$$\begin{aligned} 2\,\mathscr{U}_e{}^V &= \left(\mu+\kappa\right)\varepsilon_{ij}\,\varepsilon_{ij} + \left(\mu-\kappa\right)\varepsilon_{ji}\,\varepsilon_{ij} + \lambda\,\varepsilon_{ii}\,\varepsilon_{jj} \\ &\quad + \left(\gamma+\beta\right)\tau_{ij}\,\tau_{ij} + \left(\gamma-\beta\right)\tau_{ji}\,\tau_{ij} + \alpha\,\tau_{ii}\,\tau_{jj}, \end{aligned} \tag{22}$$

and:

$$\begin{aligned} 2\,\mathscr{U}_e{}^V &= 2\mu\varepsilon_{ij}^{s}\,\varepsilon_{ij}^{s} + \lambda\,\varepsilon_{ii}\,\varepsilon_{jj} + 2\kappa\varepsilon_{ij}^{a}\,\varepsilon_{ij}^{a} \\ &\quad + 2\gamma\tau_{ij}^{s}\,\tau_{ij}^{s} + \alpha\,\tau_{ii}\,\tau_{jj} + 2\beta\tau_{ij}^{a}\,\tau_{ij}^{a}. \end{aligned} \tag{23}$$

The fact that the strain energy density expression should have a positive definite quadratic form imposes the following restrictions on the material elastic constants [4]:

$$\begin{aligned} &\mu > 0, \qquad \kappa > 0, \qquad 2\mu+3\lambda > 0, \\ &\gamma > 0, \qquad \beta > 0, \qquad 2\gamma+3\alpha > 0. \end{aligned} \tag{24}$$

SIMPLIFICATION TO COUPLE-STRESS ELASTICITY

Consider again the general homogeneous, isotropic, and centrally symmetric elastic body under the action of (finite) body volume force $\underset{\to}{f}^V$ and moment $\underset{\to}{m}^V$. For such a body, utilizing the linear micropolar theory of elasticity results in the relations and definitions given by Eqns. (1)–(24). Now by taking into account the constitutive relations, especially the first relation given in Eqn. (12), and letting the micropolar couple modulus κ tend to infinity while noting that the force stress tensor $\underset{\leftrightarrow}{\sigma}$ should remain finite, one can conclude that the antisymmetric part of the strain tensor, *i.e.* $\underset{\leftrightarrow}{\varepsilon}^{\mathrm{a}}$, should vanish:

$$\begin{aligned} \varepsilon^{\mathrm{a}}_{ij} &= e_{ijk}\left(\theta_k - \vartheta_k\right) = 0, \\ \vartheta_i &= \theta_i = \frac{1}{2}\, e_{ijk}\, u_{k,j}. \end{aligned} \tag{25}$$

Consequently, the kinematic relations given by Eqns. (1)–(5) can be simplified to:

$$\begin{aligned} \varepsilon_{ij} &= \varepsilon^{\mathrm{s}}_{ij} + \varepsilon^{\mathrm{a}}_{ij} = \frac{1}{2}\left(u_{j,i} + u_{i,j}\right), \\ \varepsilon^{\mathrm{s}}_{ij} &= \frac{1}{2}\left(u_{j,i} + u_{i,j}\right), \\ \varepsilon^{\mathrm{a}}_{ij} &= 0, \\ \tau_{ij} &= \tau^{\mathrm{s}}_{ij} + \tau^{\mathrm{a}}_{ij} = \frac{1}{2}\, e_{jkl}\, u_{l,ki}, \\ \tau^{\mathrm{s}}_{ij} &= \frac{1}{4}\, e_{jkl}\, u_{l,ki} + \frac{1}{4}\, e_{ikl}\, u_{l,kj}, \\ \tau^{\mathrm{a}}_{ij} &= \frac{1}{4}\, e_{jkl}\, u_{l,ki} - \frac{1}{4}\, e_{ikl}\, u_{l,kj}, \\ \tau_{ii} &= 0. \end{aligned} \tag{26}$$

For such a case (*i.e.* when $\kappa \to \infty$ and thus $\varepsilon^{\mathrm{a}}_{ij} = 0$ and $\tau_{ii} = 0$), the constitutive relations given by Eqn. (13) take the form:

$$\begin{aligned} \sigma^{\mathrm{s}}_{ij} &= 2\,\mu\,\varepsilon_{ij} + \lambda\,\varepsilon_{kk}\,\mathbb{1}_{ij}, & \sigma^{\mathrm{a}}_{ij} &= \infty \times 0 = ¿, \\ \chi^{\mathrm{s}}_{ij} &= 2\,\gamma\,\tau^{\mathrm{s}}_{ij}, & \chi^{\mathrm{a}}_{ij} &= 2\,\beta\,\tau^{\mathrm{a}}_{ij}, \end{aligned}$$

where ¿ is a symbol that represents a numerical quantity whose magnitude cannot be determined (an indeterminate quantity). However, as $\tau_{ii} = 0$ this form impose an unnecessary constraint on the couple stress tensor, that is $\chi_{ii} = 0$. To remove this constraint it can be assumed that in the constitutive relations given by Eqn. (13) (in addition to the micropolar couple modulus κ) the micropolar twist coefficient α goes to infinity as well. Using the second relation of Eqn. (14), this assumption gives rise to the following form for the constitutive relations in Eqn. (13):

$$\begin{aligned} \sigma^{\mathrm{s}}_{ij} &= 2\,\mu\,\varepsilon_{ij} + \lambda\,\varepsilon_{kk}\,\mathbb{1}_{ij}, & \sigma^{\mathrm{a}}_{ij} &= \infty \times 0 = ¿, \\ \chi^{\mathrm{s}}_{ij} - \frac{1}{3}\,\chi_{kk}\,\mathbb{1}_{ij} &= 2\,\gamma\,\tau^{\mathrm{s}}_{ij}, & \chi^{\mathrm{a}}_{ij} &= 2\,\beta\,\tau^{\mathrm{a}}_{ij}, \\ \chi_{ii} &= \infty \times 0 = ¿, \end{aligned} \tag{27}$$

or equivalently:

$$\begin{aligned} \sigma_{ij} - \sigma^{\mathrm{a}}_{ij} &= 2\,\mu\,\varepsilon_{ij} + \lambda\,\varepsilon_{kk}\,\mathbb{1}_{ij}, \\ \chi_{ij} - \frac{1}{3}\,\chi_{kk}\,\mathbb{1}_{ij} &= \left(\gamma + \beta\right)\tau_{ij} + \left(\gamma - \beta\right)\tau_{ij}, \\ \sigma^{\mathrm{a}}_{ij} = \infty \times 0 = ¿, &\qquad \chi_{ii} = \infty \times 0 = ¿. \end{aligned} \tag{28}$$

It is worthwhile to note here that the indeterminacy of the asymmetric force stress tensor $\underset{\leftrightarrow}{\sigma}^{\mathrm{a}}$ and the summation of normal couple stresses χ_{ii} means they cannot be obtained from the constitutive relations and (if possible) one should use the kinetic balance relations to determine them.

Whereas the bulk modulus B, strain Poisson's ratio ν, and Young's modulus E defined in Eqns. (14) and (15) remain unchanged as κ and α tend to infinity, by letting α in Eqns. (14) and (16) go to infinity one can derive the tortile bulk modulus $\mathscr{B}$, twist Poisson's ratio ξ, and tortile modulus $\mathscr{E}$ which correspond to the current case, that is:

$$\begin{aligned} B &= \lambda + \frac{2}{3}\,\mu, \qquad \nu = \frac{\lambda}{2\left(\mu + \lambda\right)}, \\ E &= \frac{\mu\left(2\,\mu + 3\,\lambda\right)}{\mu + \lambda} = 2\,\mu\left(1 + \nu\right), \\ \mathscr{B} &= \infty, \qquad \xi = \frac{1}{2}, \qquad \mathscr{E} = 3\,\gamma. \end{aligned} \tag{29}$$

By letting κ and α go to infinity, however, the kinetic relations will remain unchanged as no restriction is imposed on the force and couple stress tensors and, therefore, one can repeat, for example, the equilibrium relations given by Eqn. (6):

$$\begin{aligned} \sigma_{ji,j} + f^V_i &= \rho^V\,\ddot{u}_i, \\ \chi_{ji,j} + e_{ijk}\,\sigma_{jk} + m^V_i &= \iota^V\,e_{ijk}\,\ddot{u}_{k,j}, \end{aligned} \tag{30}$$

or more properly the equilibrium relations in Eqn. (10):

$$\begin{aligned} \sigma^{\mathrm{s}}_{ji,j} + \frac{1}{2}\,e_{ijk}\left(\chi_{lk,lj} + m^V_{k\,,j} - \iota^V\,e_{klm}\,\ddot{u}_{m,lj}\right) + f^V_i &= \rho^V\,\ddot{u}_i, \\ e_{ijk}\left(\chi_{lk,l} + m^V_k - \iota^V\,e_{klm}\,\ddot{u}_{m,l}\right) &= 2\,\sigma^{\mathrm{a}}_{ji}. \end{aligned} \tag{31}$$

Substitution from the constitutive relations of Eqn. (28) into the balance relations given by Eqn. (31) (or revision of the motion equations in Eqn. (19) for the case when $\kappa \to \infty$ and $\alpha \to \infty$)

results in the corresponding equations of motion:

$$\begin{aligned}
&\mu\, u_{i,jj} + \left(\mu+\lambda\right) u_{j,ji} + \frac{1}{4}\, e_{ijk} \left(\gamma+\beta\right) e_{klm}\, u_{m,lnnj} \\
&\quad + \frac{1}{2}\, e_{ijk} \left(m^{V}_{k,j} - \frac{1}{2}\, \iota^{V}\, e_{klm}\, \ddot{u}_{m,lj} \right) + f^{V}_{i} = \rho^{V}\, \ddot{u}_{i}, \\
&2\,\sigma^{\mathrm{a}}_{ji} - \frac{1}{3}\, e_{ijk}\, \chi_{ll,k} = \frac{1}{2}\, e_{ijk} \left(\gamma+\beta\right) e_{klm}\, u_{m,lnn} \\
&\quad + e_{ijk} \left(m^{V}_{k} - \frac{1}{2}\, \iota^{V}\, e_{klm}\, \ddot{u}_{m,l} \right).
\end{aligned} \tag{32}$$

Here the first relation of Eqn. (32) corresponds to a set of three PDEs, enough for determination of the displacement vector. However, the second relation of Eqn. (32) also corresponding to a set of three PDEs does not provide enough information to compute the undetermined parts of the force and couple stress tensors, *i.e.* $\underset{\leftrightarrow}{\sigma}^{\mathrm{a}}$ and χ_{ii}, from a known displacement vector. Indeed, there are only three equations that should be used to determine four unknowns (three elements of $\underset{\leftrightarrow}{\sigma}^{\mathrm{a}}$ and the scalar χ_{ii}).

Finally when $\kappa \to \infty$ and $\alpha \to \infty$, the strain energy density $\mathscr{U}_{\mathrm{e}}^{V}$ can be written as (compare this to Eqns. (20) and (21) given previously for a general micropolar case):

$$\begin{aligned}
2\,\mathscr{U}_{\mathrm{e}}^{V} &= \left(\sigma_{ij} - \sigma^{\mathrm{a}}_{ij}\right) \varepsilon_{ij} + \left(\chi_{ij} - \frac{1}{3}\,\chi_{kk}\,\mathbb{1}_{ij}\right) \tau_{ij} \\
&= \sigma^{\mathrm{s}}_{ij}\,\varepsilon_{ij} + \left(\chi^{\mathrm{s}}_{ij} - \frac{1}{3}\,\chi_{kk}\,\mathbb{1}_{ij}\right) \tau^{\mathrm{s}}_{ij} + \chi^{\mathrm{a}}_{ij}\,\tau^{\mathrm{a}}_{ij}.
\end{aligned} \tag{33}$$

Now substituting from Eqn. (27) or (28) into Eqn. (33) results in the following expression for strain energy density $\mathscr{U}_{\mathrm{e}}^{V}$ (in comparison with Eqn. (22) or (23)):

$$\begin{aligned}
2\,\mathscr{U}_{\mathrm{e}}^{V} &= 2\,\mu\varepsilon_{ij}\,\varepsilon_{ij} + \lambda\,\varepsilon_{ii}\,\varepsilon_{jj} + \left(\gamma+\beta\right) \tau_{ij}\,\tau_{ij} + \left(\gamma-\beta\right) \tau_{ji}\,\tau_{ij} \\
&= 2\,\mu\varepsilon_{ij}\,\varepsilon_{ij} + \lambda\,\varepsilon_{ii}\,\varepsilon_{jj} + 2\,\gamma\,\tau^{\mathrm{s}}_{ij}\,\tau^{\mathrm{s}}_{ij} + 2\,\beta\,\tau^{\mathrm{a}}_{ij}\,\tau^{\mathrm{a}}_{ij},
\end{aligned} \tag{34}$$

which have a positive definite form provided (compared to the conditions in Eqn. (24)):

$$\begin{aligned}
&\mu > 0, \qquad 2\,\mu + 3\,\lambda > 0, \\
&\gamma > 0, \qquad \beta > 0.
\end{aligned} \tag{35}$$

The relations given by Eqns. (25)–(35) (especially after ignoring the terms containing the material microinertia density ι^{V}) are known as the relations of the indeterminate couple-stress theory [11] (since, as mentioned previously, the number of equations in Eqn. (32) are not enough for complete determination of engaged unknowns, the couple-stress theory is usually called the indeterminate couple-stress theory).

To sum up, one can conclude that the couple-stress theory with four material elastic constants μ, λ, γ, and β is a special case of the more general micropolar theory of elasticity consisting of six material elastic constants μ, κ, λ, γ, β, and α which can be obtained mathematically as $\kappa \to \infty$ and $\alpha \to \infty$ (it is also usual to neglect the microinertia effects by assuming $\iota^{V} \to 0$).

SIMPLIFICATION TO CLASSICAL ELASTICITY

Consider the relations of the couple-stress theory, given in the previous section by *i.e.* Eqns. (25)–(35), which were obtained from the relations of the micropolar elasticity theory by assuming $\kappa, \alpha \to \infty$. By taking another step and letting the micropolar twist coefficients γ and β and the microinertia density ι^{V} go to zero, the couple-stress theory relations will further simplify to a set of relations in which the effects of the couple stresses are almost negligible. For such a case the constitutive relations will be:

$$\begin{aligned}
&\sigma_{ij} - \sigma^{\mathrm{a}}_{ij} = 2\,\mu\,\varepsilon_{ij} + \lambda\,\varepsilon_{kk}\,\mathbb{1}_{ij}, \\
&\chi_{ij} - \frac{1}{3}\,\chi_{kk}\,\mathbb{1}_{ij} = 0, \\
&\sigma^{\mathrm{a}}_{ij} = ¿, \qquad \chi_{ii} = ¿.
\end{aligned}$$

Although one can continue while keeping the indeterminate portion of the couple stress, *i.e.* χ_{ii}, it is more useful to neglect the couple stresses completely and consequently write the constitutive relations as:

$$\begin{aligned}
\sigma_{ij} - \sigma^{\mathrm{a}}_{ij} &= 2\,\mu\,\varepsilon_{ij} + \lambda\,\varepsilon_{kk}\,\mathbb{1}_{ij}, \\
\sigma^{\mathrm{a}}_{ij} &= ¿, \\
\chi_{ij} &= 0.
\end{aligned} \tag{36}$$

This implies that the couple stresses do not exist and as a result there is no need to define the twist tensor and derive the equations related to it. One can accordingly simplify Eqns. (25)–(35) to obtain; the definitions of the strains and the micro or macro rotations:

$$\begin{aligned}
\varepsilon_{ij} &= \frac{1}{2}\left(u_{j,i} + u_{i,j}\right), \\
\vartheta_{i} = \theta_{i} &= \frac{1}{2}\, e_{ijk}\, u_{k,j},
\end{aligned} \tag{37}$$

the constitutive relations:

$$\sigma^{\mathrm{s}}_{ij} = \sigma_{ij} - \sigma^{\mathrm{a}}_{ij} = 2\,\mu\,\varepsilon_{ij} + \lambda\,\varepsilon_{kk}\,\mathbb{1}_{ij}, \qquad \sigma^{\mathrm{a}}_{ij} = ¿, \tag{38}$$

the definitions of the bulk modulus, strain Poisson's ratio, and Young's modulus:

$$B=\lambda+\frac{2}{3}\mu, \qquad \nu=\frac{\lambda}{2\left(\mu+\lambda\right)},$$
$$E=\frac{\mu\left(2\mu+3\lambda\right)}{\mu+\lambda}=2\mu\left(1+\nu\right), \tag{39}$$

the relations of the balance of linear and angular momenta:

$$\sigma_{ji,j}+f_i^V=\rho^V\ddot{u}_i,$$
$$e_{ijk}\sigma_{jk}+m_i^V=0, \tag{40}$$

$$\sigma^{\mathrm{s}}_{ji,j}+f_i^V+\frac{1}{2}e_{ijk}m^V_{k},_j=\rho^V\ddot{u}_i,$$
$$e_{ijk}m_k^V=2\sigma^{\mathrm{a}}_{ji}, \tag{41}$$

$$\mu u_{i,jj}+\left(\mu+\lambda\right)u_{j,ji}+\frac{1}{2}e_{ijk}m^V_{k},_j+f_i^V=\rho^V\ddot{u}_i,$$
$$2\sigma^{\mathrm{a}}_{ji}=e_{ijk}m_k^V, \tag{42}$$

the strain energy density definitions:

$$2\mathscr{U}_{\mathrm{e}}{}^V=\left(\sigma_{ij}-\sigma^{\mathrm{a}}_{ij}\right)\varepsilon_{ij}=\sigma^{\mathrm{s}}_{ij}\varepsilon_{ij}, \tag{43}$$

$$2\mathscr{U}_{\mathrm{e}}{}^V=2\mu\varepsilon_{ij}\varepsilon_{ij}+\lambda\varepsilon_{ii}\varepsilon_{jj}, \tag{44}$$

and the conditions under which the strain energy has a positive definite quadratic form:

$$\mu>0, \qquad 2\mu+3\lambda>0. \tag{45}$$

The set of relations given by Eqns. (37)–(45) correspond to the asymmetric theory of classical elasticity [12]. Compared to the (well-known) symmetric classical elasticity, in the asymmetric theory of classical elasticity, although the strain tensor is symmetric, the force stress tensor can be asymmetric in the presence of a volume moment distribution. There is no constitutive relation for the antisymmetric part of the stress tensor (as given by Eqn. (38)) and the antisymmetric stress tensor is instead determined by the angular momentum balance equation (as given by Eqn. (41) or (42)). Also, the volume moment distribution appears as an equivalent force distribution in the linear momentum balance equation (as in Eqn. (41) or (42)).

Accordingly, one can conclude that the asymmetric classical theory of elasticity with two material elastic constants μ and λ is a special case of the more general couple-stress theory including four material elastic constants μ, λ, γ, and β which can be obtained mathematically as $\gamma\to0$, $\beta\to0$, and $\iota^V\to0$ [13].

Recalling that the couple-stress theory is itself a special case of the micropolar theory of elasticity, one can obtain the classical theory of elasticity directly from the micropolar theory of elasticity by letting $\kappa,\alpha\to\infty$ and $\gamma,\beta,\iota^V\to0$. In other words, the couple-stress theory is an intermediate theory derived through the process of recovering the classical elasticity theory from the micropolar elasticity theory. This is summarized in the flowchart shown in Fig. 3.

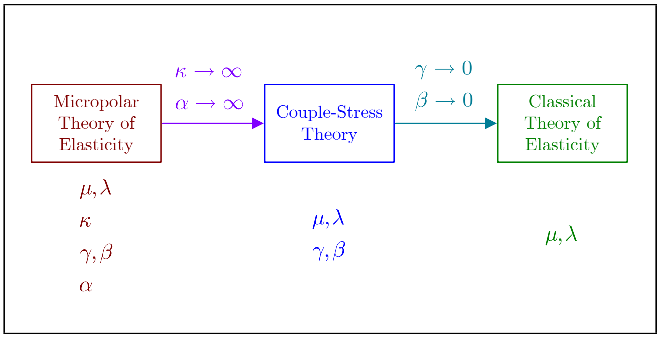

FIGURE 3. SEQUENTIAL STRUCTURE FOR SIMPLIFICATION OF THE MICROPOLAR ELASTICITY THEORY TO THE COUPLE-STRESS AND CLASSICAL ELASTICITY THEORIES.

SUMMARY AND CONCLUSIONS

The controversial nature of the well-developed theory of micropolar elasticity is a drawback for this more elaborate and comprehensive material model. This controversy is mainly about the relationships between the micropolar elasticity theory and the classical elasticity theory and their corresponding material elastic constants. To be specific , the micropolar elastic model with zero micropolar elastic constants including a zero couple modulus κ, which is traditionally known to coincide with the classical elastic model, bears (physical) difficulties [8].

One should note that the micropolar couple modulus κ determines the strength of coupling between the displacement and local rotation fields [14]. Though, simplifying the micropolar elasticity for the case $\kappa = 0$ is more straightforward, this corresponds to a decoupling of the rotational and translational degrees of freedom (DOFs) [15]. Therefore, a micropolar elasticity model with $\kappa = 0$ corresponds to an elastic continuum in which the constitutive particles or cells are free to rotate and indeed in the presence of a volume moment rotate infinitely (a singularity occurs in the presence of a volume moment).

This paper presented an alternative approach for a step-by-step simplification of the micropolar elasticity model to the classical elasticity model in which letting $\kappa \to 0$ was not required. Indeed, it was shown that the micropolar elasticity model with six elastic constants μ, κ, λ, γ, β, and γ will be simplified to the couple-stress elasticity model with four elastic constants μ, λ, γ, and β provided $\kappa, \alpha \to \infty$ (and usually $\iota^V \to 0$). Then letting $\gamma, \beta, \iota^V \to 0$, will further simply the model to the classical elasticity model with two elastic constants μ and λ.

The presented approach is beneficial as, first, it does not enforce a zero micropolar couple modulus, and second, it suggests a sequential relationship between the micropolar, couple-stress, and classical elasticity models (as shown in Fig. 3). Note that in the conventional approach the classical and couple-stress elasticity models corresponds to two different simplified cases of the micropolar elasticity model, respectively where $\kappa \to 0$ and $\kappa \to \infty$. This conventional approach is inconsistent with the fact that the couple-stress elasticity theory is a generalized form of the classical elasticity theory [16].

It should be noted that the presented approach and suggested conditions for recovery of the classical elasticity model from the micropolar elasticity model are consistent with those suggested in [17–19] which were obtained from a different point of view, that is considering the relationships between the structural characteristics of classical and micropolar gyroelastic materials.

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