An Introduction to List Colorings of Graphs

(ABSTRACT)

One of the most popular and useful areas of graph theory is graph colorings. A graph coloring is an assignment of integers to the vertices of a graph so that no two adjacent vertices are assigned the same integer. This problem frequently arises in scheduling and channel assignment applications. A list coloring of a graph is an assignment of integers to the vertices of a graph as before with the restriction that the integers must come from specific lists of available colors at each vertex. For a physical application of this problem, consider a wireless network. Due to hardware restrictions, each radio has a limited set of frequencies through which it can communicate, and radios within a certain distance of each other cannot operate on the same frequency without interfering. We model this problem as a graph by representing the wireless radios by vertices and assigning a list to each vertex according to its available frequencies. We then seek a coloring of the graph from these lists.

key words:list coloring- k-Choosability- Planar — Greedy Algorithm

Graphs

Introduction

Let G = (V,E) be a graph. We denote the vertex set of G by V (G) and the edge set of G byE(G). The number of vertices is denoted |V (G)| and similarly for the number of edges, |E(G)|.

A popular area of graph theory is the study of graph colorings. A proper coloring of a graph is a function f : V (G) →Z_+ such that for all u, v ∈V (G), f(u) 6= f(v) if uv∈E(G). We define the chromatic number of G, χ(G), to be the least positive integer k such that G has a proper coloring assigning the integers {1, 2, . . . , k} to V (G). Furthermore, if k is any integer such that G has a proper coloring from the colors {1, 2, ..., k}, we say that G is k-colorable.

Let C be a set of colors, and for each v ∈V (G), let L :V (G) →2^C be a function assigning to each vertex v ∈V (G) a list of colors L(v) ⊆C. If there is a function f :V (G) →C such that f(v) ∈L(v) for all v ∈V (G) and f(u) 6= f(v) for uv∈E(G), then G is said to beL-colorable [4]. This defines the notion of a list coloring. Note that a graph coloring as previously defined is a special case of list colorings, namely the case when all the lists are the sets $\{1, 2, \ldots, \chi(G)\}$.

If k is a positive integer, the function L is such that $|L(v)| = k$ for all v in V (G), and the graphG has a proper list coloring, then we say G is k-choosable, and we define the choice number, $\chi_L(G)$, to be the minimum such k so that G has a proper list coloring no matter what lists are assigned to the vertices of G. Note that k-choosable implies k-colorable, but not conversely as we will soon see. Much of the research in list colorings involves finding $\chi_L(G)$ for particular types of graphs or given various restrictions on the coloring rules. However, we can use any function $L : V(G) \to 2^C$ to assign lists to the vertices of a graph G, as we will see in Chapter 3.[1]

As noted in , since the usual graph colorings are special cases of list colorings, we have $\chi(G) \leq \chi_L(G)$ for all graphs G. Erd″os et al. proved with Figure 1.1 that strict inequalities exist. For all bipartite graphs, $\chi(G) = 2$, but in Figure 1.1, we see that each vertex has a list of length two as shown on the vertex, but there is no proper coloring possible from the assigned lists. Thus, for this graph, $\chi_L(G) > \chi(G)$. In fact, the authors also show that there is no general bound for how much $\chi_L(G)$ can exceed $\chi(G)$.

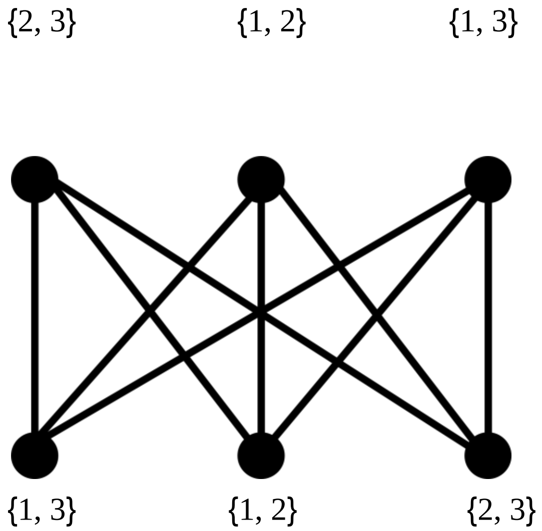

Figure 1.1: A bipartite graph that is not 2-choosable

.[1].P. Erd″os, A.L. Rubin, and H. Taylor. Choosability in graphs. In Proceedings of the West Coast Conference on Combinatorics, Graph Theory and Computing, Congres. Num. 26, pages 125–157, 1979.

Theorem 1 (Erd˝os, Rubin and Taylor[2]).There is no bound on how much $\chi_L(G)$ can exceed $\chi(G)$ as $|V(G)|$ increases.

Applications of List Colorings

Graph colorings are useful to solve various problems ranging from scheduling [2] to the channel assignment problem. List colorings in particular arise naturally from applications with restrictions on which values can be assigned to certain objects. In this section, we offer a few examples of these problems and how list colorings are used to solve them.
In [3], Zeitlhofer and Wess describe using list colorings to determine register assignments for computing processes. They discuss a situation with multiple functional units–some capable of addition and multiplication, and some capable of only addition. List coloring is then used for instruction scheduling. For example, if an operation requires a multiplication, it cannot be assigned to an addition-only unit. Thus, each operation is assigned a list of units that can process it, and a list coloring problem emerges. A second list coloring problem arises when assigning registers to store intermediate values. The operations are assigned lists of registers depending on which registers the necessary computation units access, i.e. if an operation requires an addition followed by a multiplication, the intermediate value of the addition should not be stored in a register that is not accessed by a multiplicationcapable unit. Solving these problems then determines an appropriate register assignment and instruction schedule for the desired computation. Because the list coloring problem is NP-hard, Zeitlhofer and Wess take advantage of certain properties of the graphs emerging from their registry assignment problem in order to find a coloring.

Another application of list colorings is the channel assignment problem. As Ramachandran, Belding, Almeroth, and Buddhikot discuss in [9], wireless networks near each other often interfere. Thus, to limit the interference and to satisfy hardware requirements, one must limit the frequencies available to a router. This situation is modeled as a list coloring problem. Ramachandran et al. describe assigning frequencies to a wireless mesh network built on a mixture of multi-radio and single-radio routers. Multi-radio routers have multiple

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[2].J.A. Bondy and U.S.R. Murty.Graph Theory, volume 244 of Graduate Texts in Mathematics. Springer, New York, 2008.
[3].T. Zeitlhofer and B. Wess. List-coloring of interval graphs with application to register assignment for heterogenous register-set architectures. Signal Process., 83:1411–1425, 2003.

wireless radios which can be tuned to nonoverlapping channels and communicate with severalCourtney L. Baber Chapter 1. Introduction 6
other radios. Single-radio routers, on the other hand, can only be tuned to one channel. In representing this problem as a graph theory problem, it is assumed that the hardware for the network is already in place, so that a network topology is given. Each radio then corresponds to a vertex. Thus, if a router has three radios, that router corresponds to three vertices. Each edge represents a wireless link between radios in the given network topology. Call this graph G. The authors then construct the Multi-radio Conflict Graph (MCG) where each edge in G becomes a vertex. Then if two wireless links in G interfere with each other, an edge is drawn between the vertices in MCG that represent those links. Lists of available frequencies are assigned to each vertex of MCG, and a proper coloring is sought. Ramachandran et al. cope with the difficulty of the list coloring problem by using a breadth-first search algorithm to assign channels to the vertices of MCG. Each network has a gateway router where the network connects to an external network. This router is chosen as the starting point of the breadth-first search since its connections are assumed to host the most traffic. One of the vertices of MCG corresponding to a wireless link with the gateway router will be the first colored, and the coloring will be extended to the rest of MCG in a way that minimizes interference[4]. Note that the list coloring problem presented here is actually a list coloring of the edges of G. Edge colorings will be discussed in more detail in Chapter 4.

k-Choosability of Graphs

The simplest functions to understand are constant functions. For this reason, we first analyze k-choosability, for lists of constant length k, before moving on to list functions of nonconstant
length in the next chapter. In this chapter, we completely characterize 2-choosable graphs and show that planar graphs are 5-choosable. As is often the case in graph theory, many of the proofs to follow are lengthy constructive arguments. We ask for the reader’s patience, for the results are quite beautiful, and there is much to learn from the techniques of these proofs.

2-Choosable Graphs

[4].K. Ramachandran, E. Belding, K. Almeroth, and M. Buddhikot. Interference-aware channel assignment in multi-radio wireless mesh networks. In INFOCOM 2006. 25th IEEE International Conference on Computer Communications. Proceedings, pages 1–12, 2006.

In this section, we provide a characterization of all 2-choosable graphs which can be found in[5]. First, though, we must provide a few definitions.

Recall that k-choosablemeans that the list on each vertex has length k, and that from any such set of lists, the graph G may be properly colored.

Planar Graphs

In [5], Erd˝os et al. conjectured that all planar graphs were 5-choosable. Fourteen years later, this conjecture was proven by CarstenThomassen in[6]. We present the proof here, but first we must present a definition and a lemma.

A near-triangulation is a simple planar graph consisting of an outer cycle whose interior has been divided up into triangles through the addition of vertices and edges [6].

Lemma 1.For every simple planar graph G, there is a near-triangulation that contains an isomorphic copy of G as a subgraph.

We introduce a greedy algorithm for list coloring a graph. Given a list channel assignment problem, (G,L,w), and an ordering of the vertices of G, Algorithm 1 returns a (partial) coloringc : $V(G) \rightarrow Z_+$ such that c(v) is in L(v) for all vertices v that receive colors.

Algorithm 1 Greedy Algorithm

Input: an ordering of the vertices of G, $v_1, v_2, ..., v_n$, weight function w, lists of colors $L(v_i)$
Output: a (partial) coloring c of G

color←minimum color in the lists $L(v_i)$

max←maximum color in the lists $L(v_i)$

whilecolor < max do
forj = 1 to ndo
if$v_j$is uncolored and color is in $L(v_j)$ then
iffor every colored neighbor $v_k$of v_j, $|c(v_k) - \text{color}| \geq w(v_kv_j)$ then
$c(v_j)$ = color
end if
end if
end for
color←color + 1
end while

We let the reader verify that in the case Algorithm 1 returns a full coloring of G, it is a

proper coloring. It is also straightforward to see that if Algorithm 1 does not assign the color kto a vertex v_j, then one of the following must hold:
(i) The vertex $v_j$has been assigned a color k' with k' < k.

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[5]P. Erd˝os, A.L. Rubin, and H. Taylor. Choosability in graphs. In Proceedings of the West Coast Conference on Combinatorics, Graph Theory and Computing, Congres. Num. 26, pages 125–157, 1979..

[6].C. Thomassen. Every planar graph is 5-choosable. J. of Combin. Theory Ser. B, 62(1):180–181, 1994.

(ii) The color k has been assigned to a neighbor v_i with $i< j$ in the given ordering.
(iii) Some neighbor v_i of v

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3.T. Zeitlhofer and B. Wess. List-coloring of interval graphs with application to register assignment for heterogenous register-set architectures. Signal Process., 83:1411–1425, 2003.

4. K. Ramachandran, E. Belding, K. Almeroth, and M. Buddhikot. Interference-aware channel assignment in multi-radio wireless mesh networks. In INFOCOM 2006. 25th IEEE International Conference on Computer Communications. Proceedings, pages 1–12, 2006.

5 . P. Erd˝os, A.L. Rubin, and H. Taylor. Choosability in graphs. In Proceedings of the West Coast Conference on Combinatorics, Graph Theory and Computing, Congres. Num. 26,

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