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A Potential Approach for Ordinal Games

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Abstract

We introduce the class of ordinal games with a potential, which are characterized by the absence of weak improvement cycles, the same condition used by Voorneveld and Norde (1997) for ordinal potential games.

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1 Introduction

Voorneveld and Norde (1997) proved a characterization of Ordinal Potential Games (OPG), a class of games that was introduced by Monderer and Shapley (1996). Aim of this short note is to recast the result of Voorneveld and Norde (1997) in a purely ordinal setting. It is our opinion that, in this way, their result is somehow clarified. Actually, their result can be decomposed into two parts: one is the characterization of ordinal games with a potential and the second achieves a representation by a potential, understood as a real valued function. It is only in this second step that the condition of being "properly ordered", used by Voorneveld and Norde (1997), is needed. But this condition has nothing to do with (potential) games, if we look at them from the point of view of preferences. Therefore, we focus in this paper entirely on the first part, taking this point of view as a starting point.

Moving from functions (payoffs, potentials) to preference relations has the advantage of defining the notion of ordinal games with a potential in

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what can be considered its natural setting. The definition by Monderer and Shapley (1996) only has to do with preferences as is immediately seen. This purely ordinal approach has been considered also by Kukushkin (1999), who concentrates on conditions that guarantee the existence of Nash equilibria. On the other hand, working in the setting of preferences opens the question of which kind of restrictions should be imposed upon the "potential" preference. We argue that, even if preferences of the players are total preorders, it is by no means obvious that the "potential" preferences should be total too. However, using a result of Szpilrajn (1930), we are able to show that the condition of being total can always be met.

2 Results

Let $G = (N, (X^i)_{i\in N}, (\preceq_i)_{i\in N})$ be an ordinal non-cooperative game. Here X^i are non-empty sets and $\preceq_i$ are preorders on $X = \prod_{i\in N} X^i$, that is, reflexive and transitive relations on X. Moreover, we assume that these preorders $\preceq_i$ are total.

Definition 2.1. The ordinal game $G = (N, (X^i)_{i\in N}, (\preceq_i)_{i\in N})$ is:

(i) an *ordinal game with a potential* if there exists a preorder (*potential*) $\sqsubseteq$ on X such that for every $i \in N$ and for every $x^i \in X^i, y^i \in X^i, x^{-i} \in \Pi_{j\in N\setminus\{i\}} X^j$ we have

$$(x^i, x^{-i}) \preceq_i (y^i, x^{-i}) \text{ if and only if } (x^i, x^{-i}) \sqsubseteq (y^i, x^{-i}).$$

(ii) an *ordinal game with a generalized potential* if there exists a preorder (*generalized potential*) $\sqsubseteq$ on X such that for every $i \in N$ and $x^i \in X^i, y^i \in X^i, x^{-i} \in \Pi_{j\in N\setminus\{i\}} X^j$ we have

$$(x^i, x^{-i}) \prec_i (y^i, x^{-i}) \text{ implies } (x^i, x^{-i}) \sqsubset (y^i, x^{-i}),$$

where $\prec_i$ and $\sqsubset$ denote the "strict" relations associated with $\preceq_i$ and $\sqsubseteq$ (i.e. $x \prec_i x'$ iff ($x \preceq_i x'$ and not $x' \preceq_i x$) and $x \sqsubset x'$ iff ($x \sqsubseteq x'$ and not $x' \sqsubseteq x$) for every $x, x' \in X$).

It is obvious that a game $H = (N, (X^i)_{i\in N}, (u_i)_{i\in N})$, (with $u_i : X \to \mathbb{R}$), uniquely determines an ordinal game G. If, moreover, H is an ordinal potential game according to Monderer and Shapley (1996), then G is an ordinal game with a potential in the sense of Definition 2.1. On the contrary it is easy to provide an example of an ordinal game with a potential preorder, but without any potential function: just take a one-person game with preferences that cannot be represented by a real valued utility function. Interesting in this respect is also Example 4.1 of Voorneveld and Norde (1997). This example provides a two-person game H without ordinal potential in the sense of Monderer and Shapley (1996). This game, however, has no weak improvement cycle. So, thanks to Theorem 2.2 in this paper, the induced ordinal game G has a potential preorder. What makes this example interesting for our context is that players' preferences do have a representation via (utility) functions, while this is impossible for the "potential" preference.

We did not ask $\sqsubseteq$ to be total (notice, however, that $\sqsubseteq$ is a total preorder on each set like $X^i \times \{x^{-i}\}$). When a game H has an ordinal potential function, the relation $\sqsubseteq$, induced by this potential, is also a **total** relation. The reason for not asking $\sqsubseteq$ to be total in Definition 2.1 is provided by the following example.

Example 2.1. Consider the following game H:

	L	R
T	1,1	0,0
B	0,0	2,2

Let $P(T, L) = P(B, R) = 1$ and $P(B, L) = P(T, R) = 0$. Clearly, P is an ordinal potential function for H and the induced $\sqsubseteq$ is a total relation yielding indifference between (T, L) and (B, R). But, defining instead $P'(T, L) = 2$, $P'(B, R) = 1$ and $P'(B, L) = P'(T, R) = 0$, or $P''(T, L) = 1$, $P''(B, R) = 2$ and $P''(B, L) = P''(T, R) = 0$, we get two other potential functions for which the preferences between (T, L) and (B, R) are different and reversed. We believe that this example provides enough justification for not asking that $\sqsubseteq$ is total.

It is, however, important to recognize that if we would insist on the condition that the potential $\sqsubseteq$ is a total relation, the class of ordinal games with a potential would remain the same. In order to see this it is sufficient to observe that for every ordinal game G with (preorder) potential $\sqsubseteq$, this potential can be extended to a total preorder $\sqsubseteq^*$, which still is a potential of

G. The existence of such an extension is a consequence of Szpilrajn (1930) and the construction is described in the proof of Theorem 2.1. First, the next lemma collects elementary and well known results:

Lemma 2.1. *Let X be a non-empty set and $\sqsubseteq$ a preorder on X. Define the "indifference" relation $\sim$ by*

$$x \sim y :\Longleftrightarrow (x \sqsubseteq y \text{ and } y \sqsubseteq x),$$

and the "strict" relation $\sqsubset$

$$x \sqsubset y :\Longleftrightarrow (x \sqsubseteq y \text{ and } \neg(y \sqsubseteq x)).$$

Then, $\sim$ is an equivalence relation and $\sqsubset$ is irreflexive and transitive. Furthermore:

$$(x \sqsubseteq y \text{ and } y \sqsubset z) \Longrightarrow x \sqsubset z$$

$$(x \sqsubset y \text{ and } y \sqsubseteq z) \Longrightarrow x \sqsubset z.$$

Theorem 2.1. *Let $G = (N, (X^i)_{i\in N}, (\preceq_i)_{i\in N})$ be an ordinal non-cooperative game, and let $\sqsubseteq$ be a preorder on X, which is a potential for G. Then, $\sqsubseteq$ can be extended to a total preorder $\sqsubseteq^*$ which is a potential for G.*

Proof. Consider the relation $[\sqsubseteq]$ on $X/\!\sim$ defined as:

$$[x][\sqsubseteq][y] :\Longleftrightarrow x \sqsubseteq y.$$

It is immediate that $[\sqsubseteq]$ is a (partial) order on $X/\!\sim$, i.e. a preorder which is also antisymmetric. Thanks to the theorem of Szpilrajn, in the version of Bonnet and Pouzet (1982), $[\sqsubseteq]$ can be extended to a total order $\hat{\sqsubseteq}$. Using the canonical embedding of X into $X/\!\sim$, one gets back a total preorder $\sqsubseteq^*$ that extends $\sqsubseteq$, i.e. $\sqsubseteq^*$ is such that $[\sqsubseteq^*] = \hat{\sqsubseteq}$. In order to be sure that $\sqsubseteq^*$ is still a potential for G, it is sufficient to note that $x \sqsubseteq^* y$ implies $\neg(y \sqsubset x)$. To see this, assume that $x \sqsubseteq^* y$ and $y \sqsubset x$. Because $y \sqsubset x$, we also have $y \sqsubseteq x$ and so $y \sqsubseteq^* x$ ($\sqsubseteq^*$ extends $\sqsubseteq$). So, we get $x \sim^* y$. But this is impossible since $\neg(x \sim y)$, and the definition of $\sqsubseteq^*$ guarantees that $\sim = \sim^*$. $\square$

We now pass to prove our main result, that reproduces in the ordinal setting the ideas introduced in Voorneveld and Norde (1997). First, we need the following definition.

Definition 2.2. Let $G = (N, (X^i)_{i \in N}, (\preceq_i)_{i \in N})$ be an ordinal non-cooperative game. A *path* in the strategy space X is a sequence $(x_1, x_2, \ldots)$ of elements $x_k \in X$ such that for all $k = 1, 2, \ldots$ the strategy combinations x_k and x_{k+1} differ in exactly one, say the $i(k)$-th, coordinate. A path is *non-deteriorating* if $x_k \preceq_{i(k)} x_{k+1}$ for all $k = 1, 2, \ldots$. A finite path $(x_1, \ldots, x_m)$ is called a *weak improvement cycle* if it is non-deteriorating, $x_1 = x_m$, and $x_k \prec_{i(k)} x_{k+1}$ for some $k \in \{1, \ldots, m-1\}$, where $\prec_i$ denotes the "strict" relation, associated with preorder $\preceq_i$.

Theorem 2.2. *Let $G = (N, (X^i)_{i \in N}, (\preceq_i)_{i \in N})$ be an ordinal non-cooperative game. Then G is a ordinal game with a potential if and only if there is no weak improvement cycle.*

Proof. "only if". Suppose G is an ordinal game with a potential and $\sqsubseteq$ is a potential of G. Assume that $(x_1, \ldots, x_m)$ is a weak improvement cycle. By definition, $x_k \preceq_{i(k)} x_{k+1}$ for all $k \in \{1, \ldots, m-1\}$. So, $x_k \sqsubseteq x_{k+1}$ for all $k \in \{1, \ldots, m-1\}$. Hence, $x_1 \sqsubseteq x_m$. Since for some $k \in \{1, \ldots, m-1\}$ we have that $x_k \prec_{i(k)} x_{k+1}$, we have for this k that $x_k \sqsubset x_{k+1}$, where $\sqsubset$ is the "strict" relation, associated with $\sqsubseteq$ (remember that $\sqsubseteq$ is a total preorder on $X^i \times \{x^{-i}\}$ for every x^{-i}). So, thanks to Lemma 2.1, we get $x_1 \sqsubset x_m$, contradicting the fact that $x_1 = x_m$.
"if". Suppose that there are no weak improvement cycles. Define the relation $\sqsubseteq$ on X in exactly the same way as $\triangleleft$ in Voorneveld and Norde (1997), i.e. $x \sqsubseteq y$ if and only if there is a non-deteriorating path from x to y. Let $x^i \in X^i$, $y^i \in X^i$, and $x^{-i} \in \Pi_{j \in N \setminus \{i\}} X^j$. We have to prove that

$$(x^i, x^{-i}) \preceq_i (y^i, x^{-i}) \text{ if and only if } (x^i, x^{-i}) \sqsubseteq (y^i, x^{-i}).$$

"$\Rightarrow$". Obvious from the definition of $\sqsubseteq$ (clearly, $\{(x^i, x^{-i}), (y^i, x^{-i})\}$ is a "short" non deteriorating path).
"$\Leftarrow$". Assume that $(x^i, x^{-i}) \sqsubseteq (y^i, x^{-i})$. Then, there is a non-deteriorating path $(x_1, \ldots, x_m)$ from $x_1 = (x^i, x^{-i})$ to $x_m = (y^i, x^{-i})$. If we have

$$\neg((x^i, x^{-i}) \preceq_i (y^i, x^{-i})),$$

then it is

$$(y^i, x^{-i}) \prec_i (x^i, x^{-i})$$

($\preceq_i$ is total). So, $(x_1, \ldots, x_m, x_1)$ is a weak improvement cycle. Contradiction. $\square$

Notice that if a game is an ordinal game with a potential, its preorder potential is not uniquely determined (nor in case we insist that the potential must be total). The construction used above, in the "if" part, provides the minimum preorder potential in the sense of inclusion.
We conclude stating three results, which prove that some standard features of potential games with utility functions do extend to ordinal games. Their proofs are straightforward and the first one is omitted.

Theorem 2.3. *If $G = (N, (X^i)_{i\in N}, (\preceq_i)_{i\in N})$ is an ordinal game with potential $\sqsubseteq$ then $\bar{x} \in X$ is a Nash equilibrium of G if and only if $\bar{x}$ is a Nash equilibrium for the game where preferences of all of the players are $\sqsubseteq$.*

Theorem 2.4. *If $G = (N, (X^i)_{i\in N}, (\preceq_i)_{i\in N})$ is an ordinal game with potential $\sqsubseteq$ and $\bar{x} \in X$ is maximal w.r.t. $\sqsubseteq$, then $\bar{x}$ is a Nash equilibrium for G.*

Proof. Let $x^i \in X^i$. Since $\bar{x}$ is maximal with respect to $\sqsubseteq$ we have $\neg(\bar{x} \sqsubset (x^i, \bar{x}^{-i}))$. Since $\sqsubseteq$ is total on $X^i \times \{\bar{x}^{-i}\}$ we have $(x^i, \bar{x}^{-i}) \sqsubseteq \bar{x}$ and hence $(x^i, \bar{x}^{-i}) \preceq_i \bar{x}$. □

Every preorder on a finite set has maximal elements. A consequence of Theorem 2.4 is therefore that every ordinal game with a potential and finite strategy spaces has a Nash equilibrium, a well-known result for finite potential games with utility functions.

It is also immediate to extend to the ordinal context the results of Monderer and Shapley (1996) on the characterization of games with a generalized potential by means of the "finite improvement property". We refer to Monderer and Shapley (1996) for the definition of F.I.P., and we leave to the reader its plain translation to the context of ordinal games.

Theorem 2.5. *Let $G = (N, (X^i)_{i\in N}, (\preceq_i)_{i\in N})$ be a finite ordinal game. Then, G has the F.I.P. if and only if G has a generalized (preorder) potential.*

Proof. "if": immediate adaptation of the results in Monderer and Shapley (1996). "only if": since G is finite, the preferences of the players can be represented by utility functions. The proof of Monderer and Shapley (1996) provides a function Q that induces a generalized (preorder) potential $\sqsubseteq$ on the set of strategy profiles. □

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