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General solutions for consolidation of multilayered soil with a vertical drain system

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ABSTRACT

A quasi-analytical method is newly introduced to solve the equal-strain consolidation problem of multilayered soil with a vertical drain system. Both vertical and radial drainage conditions are considered, together with the effects of drain resistance and smear. By using the method of Laplace transform with respect to time, a general explicit analytical solution for the consolidation in transformed space is obtained. Numerical inversion of the Laplace transform in the time domain is then applied to obtain the solution for calculating excess pore-water pressure. This solution is explicitly expressed and conveniently coded into a computer program for ease and efficiency of practical use. Its validity and accuracy are verified by comparing the special cases of the proposed solution with a finite-element solution and an available analytical solution. Moreover, the consolidation behavior of a four-layered soil with a vertical drain is investigated. The order of soil layers is shown to have a significant effect on the behavior of consolidation. This highlights that caution should be exercised when weighted average consolidation parameters of multilayered soil are used to analyze the consolidation behavior.

1. Introduction

Preloading is widely used to improve soft ground, and vertical drains are usually installed to accelerate the consolidation of soils in preloading (Artidteang et al., 2011; Cascone and Biondi, 2013; Chai et al., 2008, 2010; Indraratna et al., 2010, 2011, 2012; Karunaratne, 2011; Lin and Chang, 2009; Lo et al., 2008; Saowapakpiboon et al., 2009, 2010, 2011). The shear strength of soil can be increased and the post-construction foundation settlement can be eliminated significantly due to consolidation (Almeida et al., 2013; Chai and Duy, 2013; Dash and Bora, 2013; Elsawy, 2013; Rowe and Li, 2005; Rowe and Taechakumthorn, 2008). For consolidation of a homogeneous soil with a vertical drain, many analytical solutions have been proposed based on various assumptions and considerations (Abuel-Naga et al., 2012; Bari and Shahin, 2014; Barron, 1948; Castro and Sagaseta, 2013; Chai et al., 2001; Conte and Troncone, 2009; Deng et al., 2013; Geng et al., 2011, 2012; Indraratna et al., 2011; Hansbo, 1981; Hu et al., 2014; Leo, 2004; Ong et al., 2012; Onoue, 1988b; Rujikiatkamjorn and Indraratna, 2009; Tang and Onitsuka, 2000; Yoshikuni and Nakanodo, 1974; Zeng and Xie, 1989; Zhu and Yin, 2001, 2004; among others). However, natural sediments are rarely homogeneous and usually consist of several different soil layers. For consolidation of layered soil with a vertical drain, only a limited number of analytical solutions are available in literature. Xie (1995) proposed a solution for consolidation of two-layered soil with an ideal drain (i.e., without drain resistance and smear actions) by using the method of separation of variables. Tang and Onitsuka (2001) and Wang and Jiao (2004) extended this solution to include drain resistance and smear effects. On this basis, Tang et al. (2013) have recently proposed a solution for consolidation of three-layered soil. Although the solution is precise, it has been proven to have some convergence problems (Tang et al., 2013). By simply using the method of separation of variables, it is difficult to extend the solution to consolidation of soils of four or more layers. For arbitrarily multilayered soil, Rujikiatkamjorn and Indraratna (2010) proposed a solution for purely radial consolidation based on the soil slice (instead of soil element) flow continuity equations, which are similar to that developed by Hansbo (1981). Vertical drainage conditions were neglected. Nogami and Li (2003) used the matrix

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transfer method to formulate the consolidation behavior of soil with a system of horizontal thin drains and a vertical drain. However, smear effect and drain resistance were not considered in their solution. Walker and Indraratna (2009) and Walker et al. (2009) used the spectral method to obtain a solution for consolidation of multilayered soil based on a lumped governing equation derived from the soil slice flow continuity, but drain resistance was ignored. Moreover, a transcendental equation is usually involved in the available analytical solutions for consolidation of layered soil. The most difficult part of the solutions is calculating, without missing, eigenvalues or eigenvectors from the zeros of transcendental equation.

In this note, a powerful Laplace transform and its numerical inverse technique is used to formulate the consolidation behavior of arbitrarily multilayered soil with a vertical drain. The assumptions and flow continuity conditions involved are same as those applied to two-layered soil by Tang and Onitsuka (2001) and three-layered soil by Tang et al. (2013). Radial and vertical drainage conditions, as well as drain resistance and smear effects, are considered. An explicit quasi-analytical solution is derived for calculating excess pore-water pressure and the degree of consolidation. The validity and accuracy of this solution are verified against a finite-element solution for two-layered soil, and an analytical solution derived for single-layered soil by Tang and Onitsuka (2000). Moreover, the consolidation behavior of a four-layered soil with a vertical drain system is investigated.

2. Problem description

The system consisting of N contiguous homogeneous soil layers with a vertical drain is shown schematically in Fig. 1. The origin of the cylindrical coordinates is set at the top center of the vertical drain; z is positive in the downward direction; and r is in the radial direction. In Fig. 1, r_w, r_s and r_e represent the radii of the vertical drain, the smear zone and the influence zone of the vertical drain, respectively; and k_w represents the vertical hydraulic conductivity of vertical drain. Although k_w varies with the consolidation time and the confining pressure around the vertical drain (Venda Oliveira, 2013), this is beyond the scope of this study. For the ith layer of soil, h_i, k_{hi}, k_{vi}, k_{si} and m_{vi} represent, respectively, the thickness, the horizontal and vertical hydraulic conductivity of natural soil, the horizontal hydraulic conductivity of smeared soil and the coefficient of volume compressibility of the soils. The external load $q(t)$ (where t is time) is assumed to be time dependent. The top surface of the system is assumed to be pervious, and the bottom surface is impervious.

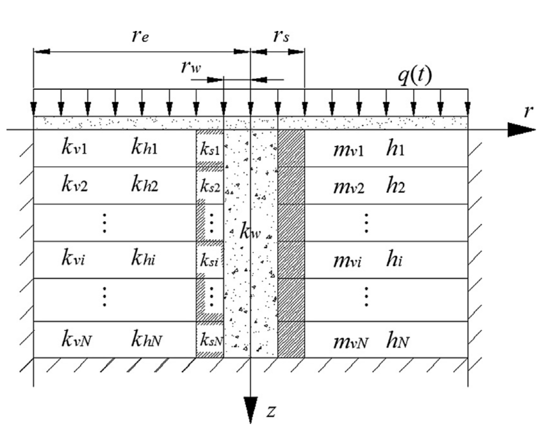

Fig. 1. Mathematical model of multilayered ground with vertical drain.

According to the assumptions made in Tang and Onitsuka (2000, 2001) and Tang et al. (2013), the following consolidation equations can be obtained.

Consolidation in the natural zone can be expressed as

$$\frac{k_{hi}}{\gamma_w}\left(\frac{1}{r}\frac{\partial u_{ni}}{\partial r}+\frac{\partial^2 u_{ni}}{\partial r^2}\right)+\frac{k_{vi}}{\gamma_w}\frac{\partial^2 \overline{u_i}}{\partial z^2}=m_{vi}\left(\frac{\partial \overline{u_i}}{\partial t}-\frac{dq}{dt}\right) \quad r_s \le r \le r_e \tag{1}$$

where $\gamma_w = 10\ \mathrm{kN/m^3}$ is the unit weight of water; $u_{ni}(r, z, t)$ is the excess pore-water pressure at an arbitrary point in the natural soil zone; $\overline{u_i}$ is the average excess pore-water pressure of soil at a given depth, as given by

$$\overline{u_i} = \frac{1}{\pi(r_e^2 - r_w^2)}\left(\int_{r_w}^{r_s} 2\pi r u_{si} dr + \int_{r_s}^{r_e} 2\pi r u_{ni} dr\right) \tag{2}$$

where $u_{si}(r, z, t)$ is the excess pore-water pressure at an arbitrary point in the smear zone.

Consolidation in the smear zone can be expressed as

$$\frac{k_{si}}{\gamma_w}\left(\frac{1}{r}\frac{\partial u_{si}}{\partial r}+\frac{\partial^2 u_{si}}{\partial r^2}\right)+\frac{k_{vi}}{\gamma_w}\frac{\partial^2 \overline{u_i}}{\partial z^2}=m_{vi}\left(\frac{\partial \overline{u_i}}{\partial t}-\frac{dq}{dt}\right) \quad r_w \le r \le r_s \tag{3}$$

Flow continuity conditions at the boundary of the vertical drain (i.e., drain resistance) can be expressed as

$$\frac{\partial^2 u_{wi}}{\partial z^2} = -\frac{2}{r_w}\frac{k_{si}}{k_w}\left(\frac{\partial u_{si}}{\partial r}\right)\bigg|_{r=r_w} \tag{4}$$

where $u_{wi}(z, t)$ is the excess pore-water pressure within the vertical drain.

The drainage conditions at the vertical boundary of the influence zone is expressed as

$$\frac{\partial u_{ni}}{\partial r}\bigg|_{r=r_e} = 0 \tag{5}$$

The drainage boundary conditions at the top and bottom surfaces of soil are listed as follows:

$$\begin{cases} u_{w1}|_{z=0} = 0 \\ \overline{u_1}\big|_{z=0} = 0 \\ \dfrac{\partial u_{wN}}{\partial z}\bigg|_{z=H_N} = 0 \\ \dfrac{\partial \overline{u_N}}{\partial z}\bigg|_{z=H_N} = 0 \end{cases} \tag{6}$$

where $H_N = \sum_{i=1}^{N} h_i$ is the whole thickness of soil.

The continuity conditions in the radial direction are as follows:

$$\begin{cases} u_{wi} = u_{si}|_{r=r_w} \\ u_{si}|_{r=r_s} = u_{ni}|_{r=r_s} \\ k_{si}\dfrac{\partial u_{si}}{\partial r}\bigg|_{r=r_s} = k_{hi}\dfrac{\partial u_{ni}}{\partial r}\bigg|_{r=r_s} \end{cases} \tag{7}$$

The continuity of excess pore-water pressure and the continuity of flow rate at the interfaces between adjoining soil layers can be expressed as:

$$\begin{cases} u_{wi}\big|_{z=H_i} = u_{w(i+1)}\big|_{z=H_i} \\ \dfrac{\partial u_{wi}}{\partial z}\Big|_{z=H_i} = \dfrac{\partial u_{w(i+1)}}{\partial z}\Big|_{z=H_i} \\ \overline{u_i}\big|_{z=H_i} = \overline{u_{i+1}}\big|_{z=H_i} \\ k_{vi}\dfrac{\partial \overline{u_i}}{\partial z}\Big|_{z=H_i} = k_{v(i+1)}\dfrac{\partial \overline{u_{i+1}}}{\partial z}\Big|_{z=H_i} \end{cases} \quad i = 1,2,\cdots,N-1 \tag{8}$$

$$\begin{cases} x_{1i} = \sqrt{\dfrac{\left(s\dfrac{c_{h1}H_N^2}{c_{vi}r_e^2}+\dfrac{c_{hi}}{c_{vi}}\dfrac{2H_N^2}{r_e^2F_i}+\varphi_i\right)+\sqrt{\left(s\dfrac{c_{h1}H_N^2}{c_{vi}r_e^2}+\dfrac{c_{hi}}{c_{vi}}\dfrac{2H_N^2}{r_e^2F_i}+\varphi_i\right)^2-4\varphi_i s\dfrac{c_{h1}}{c_{vi}}\dfrac{H_N^2}{r_e^2}}}{2}} \\ x_{2i} = \sqrt{\dfrac{\left(s\dfrac{c_{h1}H_N^2}{c_{vi}r_e^2}+\dfrac{c_{hi}}{c_{vi}}\dfrac{2H_N^2}{r_e^2F_i}+\varphi_i\right)-\sqrt{\left(s\dfrac{c_{h1}H_N^2}{c_{vi}r_e^2}+\dfrac{c_{hi}}{c_{vi}}\dfrac{2H_N^2}{r_e^2F_i}+\varphi_i\right)^2-4\varphi_i s\dfrac{c_{h1}}{c_{vi}}\dfrac{H_N^2}{r_e^2}}}{2}} \end{cases} \tag{14}$$

where $H_i = \sum_{k=1}^{i} h_k$ is the thickness of i-layered soil.

The initial conditions of the system are as follows:

$$\begin{cases} \overline{u_i}(z,0) = q(0) \\ u_{wi}(z,0) = q(0) \end{cases} \quad i = 1,2,\cdots,N \tag{9}$$

3. Laplace transforms

3.1. Governing equation

After some mathematical processing (Tang and Onitsuka, 2000, 2001), the partial differential equations containing only u_{wi} and the relationship between u_{wi} and $\overline{u_i}$ are obtained as follows:

$$c_{vi}\frac{\partial^4 u_{wi}}{\partial z^4}-\frac{\partial^3 u_{wi}}{\partial z^2\partial t}-c_{hi}\frac{2}{r_e^2F_i}\left[1+\frac{k_{vi}}{k_w}\left(n^2-1\right)\right]\frac{\partial^2 u_{wi}}{\partial z^2} + \frac{2}{r_e^2F_i}\frac{k_{hi}}{k_w}\left(n^2-1\right)\left(\frac{\partial u_{wi}}{\partial t}-\frac{dq}{dt}\right) = 0 \tag{10}$$

$$\frac{\partial^2 u_{wi}}{\partial z^2} = -\left(n^2-1\right)\frac{2}{r_e^2F_i}\frac{k_{hi}}{k_w}\left(\overline{u_i}-u_{wi}\right) \tag{11}$$

where $F_i = [\ln(n/s)+(k_{hi}/k_{si})\ln s-3/4]n^2/(n^2-1)+s^2(1-k_{hi}/k_{si})(1-s^2/4/n^2)/(n^2-1)+(k_{hi}/k_{si})(1-1/4/n^2)/(n^2-1)$; $n = r_e/r_w$; $s = r_s/r_w$; $c_{vi} = k_{vi}/\gamma_w/m_{vi}$; and $c_{hi} = k_{hi}/\gamma_w/m_{vi}$.

By letting $\phi_i = 2H_N^2(k_{hi}/k_w)(n^2-1)/(r_e^2F_i)$, $Z = z/H_N$ and $T = c_{h1}t/r_e^2$, the Laplace transform of Eq. (10) with respect to the time factor T can be derived, as given by

$$\frac{\partial^4\widetilde{u_{wi}}}{\partial Z^4}-\left(s\frac{c_{h1}H_N^2}{c_{vi}r_e^2}+\frac{c_{hi}}{c_{vi}}\frac{2H_N^2}{r_e^2F_i}+\varphi_i\right)\frac{\partial^2\widetilde{u_{wi}}}{\partial Z^2}+\varphi_i\frac{c_{h1}}{c_{vi}}\frac{H_N^2}{r_e^2}\left(s\widetilde{u_{wi}}-s\tilde{q}\right) = 0 \tag{12}$$

where $\widetilde{u_{wi}}(s) = L(u_{wi}) = \int_0^\infty u_{wi}e^{-sT}dT$ is the Laplace transform of u_{wi}; L denotes the Laplace transform; s is a complex number representing the frequency domain or Laplace-space variable; and $\tilde{q}(s) = L(q) = \int_0^\infty q(T)e^{-sT}dT$ is the Laplace transform of $q(T)$.

The general solution for Eq. (12) is

$$\widetilde{u_{wi}} = A_{1i}\exp(x_{1i}Z)+A_{2i}\exp(-x_{1i}Z)+A_{3i}\exp(x_{2i}Z) + A_{4i}\exp(-x_{2i}Z)+\tilde{q} \tag{13}$$

where A_{1i}, A_{2i}, A_{3i} and A_{4i} are coefficients to be determined; and

The Laplace transform of Eq. (11) with respect to the time factor T is given by

$$\frac{\partial^2\widetilde{u_{wi}}}{\partial Z^2} = -\varphi_i\left(\widetilde{\overline{u_i}}-\widetilde{u_{wi}}\right) \tag{15}$$

where $\widetilde{\overline{u_i}}(s) = L(\overline{u_i}) = \int_0^\infty \overline{u_i}e^{-sT}dT$ is the Laplace transform of $\overline{u_i}$.

Substituting Eq. (13) into Eq. (15) yields

$$\widetilde{\overline{u_i}} = A_{1i}\left(1-\frac{x_{1i}^2}{\varphi_i}\right)\exp(x_{1i}Z)+A_{2i}\left(1-\frac{x_{1i}^2}{\varphi_i}\right)\exp(-x_{1i}Z) + A_{3i}\left(1-\frac{x_{2i}^2}{\varphi_i}\right)\exp(x_{2i}Z)+A_{4i}\left(1-\frac{x_{2i}^2}{\varphi_i}\right)\exp(-x_{2i}Z)+\tilde{q} \tag{16}$$

3.2. Boundary conditions

The Laplace transform of Eq. (6) with respect to the time factor T is given by

$$\begin{cases} \widetilde{u_{w1}}\big|_{Z=0} = 0 \\ \widetilde{\overline{u_1}}\big|_{Z=0} = 0 \\ \dfrac{\partial\widetilde{u_{wN}}}{\partial Z}\Big|_{Z=1} = 0 \\ \dfrac{\partial\widetilde{\overline{u_N}}}{\partial Z}\Big|_{Z=1} = 0 \end{cases} \tag{17}$$

3.3. Continuity conditions at the soil interfaces

The Laplace transform of Eq. (8) with respect to time T is given by

$$\begin{cases} \left.\widetilde{u_{wi}}\right|_{Z=H_i/H_N} = \left.\widetilde{u_{w(i+1)}}\right|_{Z=H_i/H_N} \\ \left.\dfrac{\partial \widetilde{u_{wi}}}{\partial Z}\right|_{Z=H_i/H_N} = \left.\dfrac{\partial \widetilde{u_{w(i+1)}}}{\partial Z}\right|_{Z=H_i/H_N} \\ \left.\widetilde{\overline{u_i}}\right|_{Z=H_i/H_N} = \left.\widetilde{\overline{u_{i+1}}}\right|_{Z=H_i/H_N} \\ \left.k_{vi}\dfrac{\partial \widetilde{\overline{u_i}}}{\partial Z}\right|_{Z=H_i/H_N} = \left.k_{v(i+1)}\dfrac{\partial \widetilde{\overline{u_{i+1}}}}{\partial Z}\right|_{Z=H_i/H_N} \end{cases} \quad i = 1,2,\ldots,N-1 \tag{18}$$

4. Solving procedures

By substituting Eqs. (13) and (16) into the boundary conditions as given by Eq. (17), the following matrix can be obtained.

$$\begin{cases} \mathbf{T}\times\mathbf{Y}_1 = \mathbf{P}_1 \\ \mathbf{B}\times\mathbf{Y}_N = \mathbf{P}_N \end{cases} \tag{19}$$

where $\mathbf{T} = \left[\dfrac{1}{1-x_{11}^2/\varphi_1} \quad \dfrac{1}{1-x_{11}^2/\varphi_1} \quad \dfrac{1}{1-x_{21}^2/\varphi_1} \quad \dfrac{1}{1-x_{21}^2/\varphi_1}\right]$; $\mathbf{Y1} = [A_{11}\ A_{21}\ A_{31}\ A_{41}]^T$; $\mathbf{P1} = [-\tilde{q}\ -\tilde{q}]^T$;

$$\mathbf{B} = \begin{bmatrix} x_{1N}\exp(x_{1N}) & x_{1N}\exp(x_{1N})(1-x_{1N}^2/\varphi_N) \\ -x_{1N}\exp(-x_{1N}) & -x_{1N}\exp(-x_{1N})(1-x_{1N}^2/\varphi_N) \\ x_{2N}\exp(x_{2N}) & x_{2N}\exp(x_{2N})(1-x_{2N}^2/\varphi_N) \\ -x_{2N}\exp(-x_{2N}) & -x_{2N}\exp(-x_{2N})(1-x_{2N}^2/\varphi_N) \end{bmatrix}^T;$$

$\mathbf{Y}_N = [A_{1N}\ A_{2N}\ A_{3N}\ A_{4N}]^T$; and $\mathbf{P}_N = [0\ 0]^T$.

By substituting Eqs. (13) and (16) into the continuity conditions as given by Eq. (18), the following matrix can be obtained.

$$[\mathbf{C}_i]_{4\times4}\times\mathbf{Y}_i = [\mathbf{D}_i]_{4\times4}\times\mathbf{Y}_{(i+1)} \quad i = 1,2,\ldots,N-1 \tag{20}$$

where, $\mathbf{Y}_i = [A_{1i}\ A_{2i}\ A_{3i}\ A_{4i}]^T$; and details of $\mathbf{C}_i$ and $\mathbf{D}_i$ are listed in the Appendix A.

Combining Eqs. (19) and (20) yields

$$\begin{bmatrix} \mathbf{T} & & & & & & \\ \mathbf{C}_1 & -\mathbf{D}_1 & & & & & \\ & \ddots & \ddots & & & & \\ & & \mathbf{C}_i & -\mathbf{D}_i & & & \\ & & & \ddots & \ddots & & \\ & & & & \mathbf{C}_{N-1} & -\mathbf{D}_{N-1} & \\ & & & & & & \mathbf{B} \end{bmatrix}_{4N\times4N} \begin{Bmatrix} \mathbf{Y}_1 \\ \vdots \\ \mathbf{Y}_i \\ \vdots \\ \mathbf{Y}_N \end{Bmatrix}_{4N\times1} = \begin{Bmatrix} -\tilde{q} \\ -\tilde{q} \\ 0 \\ \vdots \\ 0 \end{Bmatrix}_{4N\times1} \tag{21}$$

The coefficients A_{1i}, A_{2i}, A_{3i}, and A_{4i} can be obtained by solving Eq. (21). Once these values are known, the Laplace-transformed excess pore-water pressure $\widetilde{\overline{u_i}}$ can be obtained from Eq. (16). By inverting Eq. (16), the excess pore-water pressure can be derived as follows:

$$\overline{u_i}(z,t) = \overline{u_i}(Z,T) = L^{-1}\left[\widetilde{\overline{u_i}}\ (Z,s)\right] = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty}\widetilde{\overline{u_i}}\ (Z,s)e^{sT}\mathrm{d}s = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty}\widetilde{\overline{u_i}}\ \left(\frac{z}{H_N},s\right)e^{s\frac{c_{h1}t}{r_e^2}}\mathrm{d}s \tag{22}$$

where L^{-1} denotes the inverse Laplace transform; $j = \sqrt{-1}$; and σ is such that the contour of integration is to the right-hand side of any singularities of $\widetilde{\overline{u_i}}$.

Analytical inversion of Laplace transforms can be obtained for the simplest case of one layer soil with a vertical drain system (as shown later). However, it is difficult for multilayered soil with a vertical drain system to achieve. There are several algorithms available for the numerical inversion of Laplace transforms. In this study, the De Hoog algorithm (De Hoog et al., 1982) was adopted for numerical inversion of Laplace transforms. It has been proved to be an effective and efficient technique for numerical inversion of Laplace transforms (Boupha et al., 2004; Liu and Lei, 2013).

5. Verification

5.1. Comparisons with finite-element solution

It can be seen from Eq. (2) that an averaging of the radial variation of the excess pore-water pressure is used to derive the solution. To demonstrate whether this radial averaging is a reasonable one or not, consolidation of a two-layered soil with a vertical drain is analyzed by using the proposed solution and the finite-element modeling (FEM). The calculation parameters presented by Chai et al. (2001) are adopted as follows: $r_w = 0.025$ m, $r_s = 0.15$ m, $r_e = 1.0$ m, $k_h/k_s = 5$, $k_w = 16.2\times10^{-4}$ m/s, and other soil parameters as given in Table 1. An external load $q_0 = 100$ kPa is instantaneously applied. Fig. 2 shows a side view of axisymmetrical finite-element meshes. Table 2 shows the calculated excess pore-water pressure for a time factor $T = 7.4437$ for all the meshes. The data in the last three rows of Table 2 represent respectively the radial average values of excess pore-water pressure at a given depth calculated from FEM and from the proposed solution, and their difference. It can be seen that the radial average values obtained from the proposed solution are in reasonable agreement with those obtained from the FEM. The maximum difference is less than 3 kPa. Fig. 3 shows the variations of radial average excess pore-water pressure with depth for $T = 37.2185$ and $T = 7.4437$. The results calculated from the proposed solution are also in reasonable agreement with the FEM. The validity and accuracy of the radial averaging were also verified by Tang and Onitsuka (2001) by comparing the degrees of consolidation calculated from their analytical solution for two-

Table 1
Parameters of a two-layered soil adopted from Chai et al. (2001).

Layer no. i	h_i (m)	E_i (kPa)	k_{hi} (m/s)	k_{vi} (m/s)
1	3.0	8000	4.0×10^{-8}	2.0×10^{-8}
2	7.0	4000	2.0×10^{-8}	1.0×10^{-8}

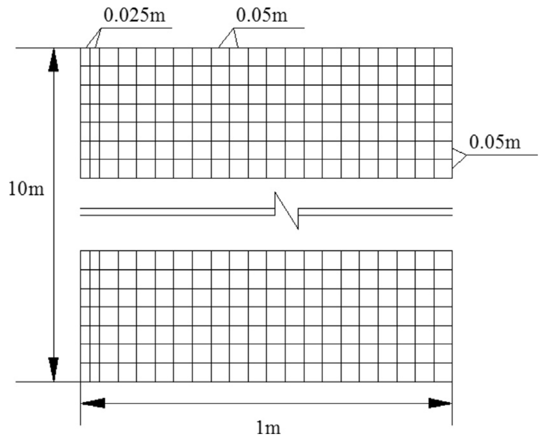

Fig. 2. Element mesh.

layered soil, and the finite-difference and finite-element modeling.

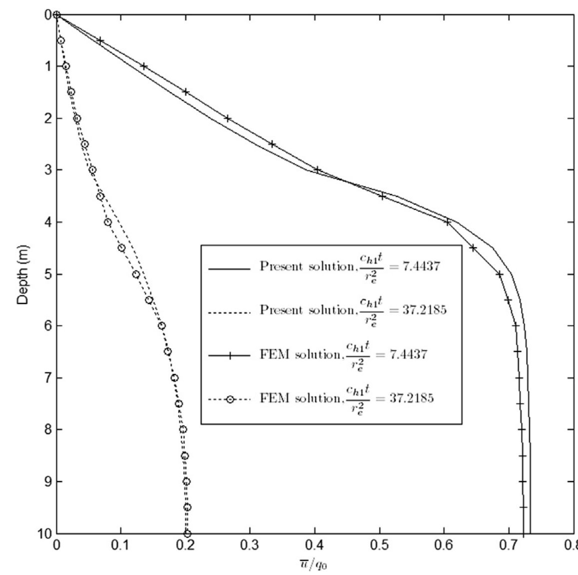

Fig. 3. Excess pore-water pressure isochrones for a two-layered soil calculated using the numerical inversion of Laplace transform and FEM.

5.2. Analytical inversion of the laplace transforms for a single layer system

By applying $N = 1$, and applying a ramp loading to the system, a solution specific to the consolidation of single-layered soil with a vertical drain system under ramp loading can be developed. The single ramp load is expressed as:

$$q(t) = \begin{cases} q_u t/t_c & 0 \le t \le t_c \\ q_u & t_c \le t \end{cases} \tag{23}$$

where q_u is the ultimate load; and t_c is the time taken to reach q_u.

Under this situation, the following coefficients can be obtained from Eq. (19) as

$$\begin{aligned} A_1 &= \frac{-\tilde{q}}{\exp(2x_1)+1}\frac{x_2^2}{x_2^2-x_1^2} \\ A_2 &= \frac{-\tilde{q}}{\exp(2x_1)+1}\frac{x_2^2}{x_2^2-x_1^2}\exp(2x_1) \\ A_3 &= \frac{-\tilde{q}}{\exp(2x_2)+1}\frac{x_1^2}{x_1^2-x_2^2} \\ A_4 &= \frac{-\tilde{q}}{\exp(2x_2)+1}\frac{x_1^2}{x_1^2-x_2^2}\exp(2x_2) \end{aligned} \tag{24}$$

where H is the thickness of the soil; and

Table 2
Excess pore-water pressure (in kPa) calculated from FEM and the proposed solution at $T = 7.4437$.

r (m)	Depth (m)									
	1	2	3	4	5	6	7	8	9	10
0.05	7.80	15.23	22.99	33.79	39.29	41.34	41.89	42.53	42.85	42.96
0.10	10.77	21.02	31.89	47.46	54.34	56.60	57.12	57.65	57.91	58.00
0.15	12.76	24.91	37.88	56.66	64.46	66.86	67.36	67.83	68.04	68.12
0.20	13.11	25.59	38.93	58.27	66.23	68.66	69.15	69.61	69.81	69.88
0.25	13.38	26.11	39.73	59.50	67.58	70.02	70.52	70.97	71.16	71.23
0.30	13.59	26.53	40.37	60.48	68.66	71.12	71.61	72.05	72.24	72.31
0.35	13.77	26.87	40.90	61.28	69.54	72.01	72.51	72.94	73.12	73.19
0.40	13.91	27.15	41.33	61.95	70.28	72.76	73.25	73.68	73.86	73.92
0.45	14.04	27.39	41.70	62.52	70.90	73.39	73.88	74.30	74.48	74.54
0.50	14.14	27.59	42.01	62.99	71.43	73.92	74.41	74.83	75.01	75.06
0.55	14.23	27.76	42.28	63.40	71.87	74.38	74.86	75.28	75.45	75.51
0.60	14.30	27.91	42.50	63.74	72.25	74.76	75.25	75.66	75.83	75.88
0.65	14.37	28.03	42.69	64.03	72.56	75.08	75.56	75.98	76.15	76.20
0.70	14.42	28.13	42.84	64.27	72.83	75.34	75.83	76.24	76.41	76.45
0.75	14.45	28.21	42.97	64.46	73.04	75.56	76.04	76.45	76.62	76.67
0.80	14.49	28.28	43.07	64.61	73.20	75.73	76.21	76.62	76.79	76.83
0.85	14.52	28.33	43.14	64.73	73.33	75.86	76.34	76.75	76.91	76.96
0.90	14.53	28.36	43.20	64.80	73.42	75.94	76.43	76.83	77.00	77.04
0.95	14.54	28.38	43.23	64.85	73.47	75.99	76.48	76.88	77.05	77.09
1.00	14.55	28.39	43.24	64.87	71.98	76.01	76.50	76.90	77.07	77.11
Radial average from FEM	13.58	26.51	40.35	60.43	68.53	71.07	71.56	72.00	72.19	72.25
Proposed solution	11.53	23.87	38.52	61.99	70.36	72.43	72.96	73.20	73.33	73.37
Difference	2.05	2.64	1.83	−1.56	−1.83	−1.36	−1.40	−1.20	−1.14	−1.12

$$\tilde{q}(s) = \begin{cases} q_u/(s^2 T_c) & 0 \le T \le T_c \\ q_u/s & T_c \le T \end{cases} \tag{25}$$

where $T_c = c_{h1} t_c / r_e^2$.

Substituting Eq. (24) into Eq. (16) yields

$$\tilde{\bar{u}} = \frac{-\tilde{q}}{\exp(2x_1)+1} \frac{x_2^2}{x_2^2 - x_1^2}\left(1 - \frac{x_1^2}{\varphi}\right)[\exp(x_1 Z) + \exp(2x_1)\exp(-x_1 Z)] + \frac{-\tilde{q}}{\exp(2x_2)+1} \frac{x_1^2}{x_1^2 - x_2^2}\left(1 - \frac{x_2^2}{\varphi}\right) \times [\exp(x_2 Z) + \exp(2x_2)\exp(-x_2 Z)] + \tilde{q} \tag{26}$$

After some mathematical processing, the excess pore-water pressure in the time domain can be derived (see the Appendix B), as

$$\bar{u} = L^{-1}\left(\tilde{\bar{u}}\right) = \begin{cases} \dfrac{q_u}{t_c}\displaystyle\sum_{k=1}^{\infty} \frac{r_e^2}{-s_k c_h} \frac{2}{M} \sin\left(\frac{Mz}{H}\right)\left(1 - \exp^{s_k c_h t / r_e^2}\right) & 0 \le t \le t_c \\ \dfrac{q_u}{t_c}\displaystyle\sum_{k=1}^{\infty} \frac{r_e^2}{-s_k c_h} \frac{2}{M} \sin\left(\frac{Mz}{H}\right)\left[\exp^{s_k (t-t_c) c_h / r_e^2} - \exp^{s_k t c_h / r_e^2}\right] & t_c \le t \end{cases} \tag{27}$$

where $M = (2k-1)\pi/2$, $k = 1,2\ldots$.

This solution is the same as that presented by Tang and Onitsuka (2000).

6. Discussion

Because of the complexity of solution to consolidation of multilayered soil with a vertical drain system, some approximate solution has been presented. The most common practice is to replace the multilayered ground with a single homogeneous soil having a thickness equaling to the total thickness of original multilayered ground and layer-thickness-weighted mean values of consolidation parameters. Onoue (1988a) pointed out that replacement using homogeneous soil having average soil constants is improper for predicting the consolidation of multilayered soils, whose error can be larger than 10%. He developed an approximate calculation method for the consolidation of multilayered soils with a vertical drain system. However, the order of soil layer in the ground is not taken into account in his method. The following example is intended to illustrate the significant effect that variations in order of soil layer in the multilayered ground can have on the behavior of consolidation.

The four idealized soil profiles (Fig. 4) all consist of four soil layers of equal depth. Instantaneous loading is applied to these multilayered soils with a vertical drain system, where $r_w = 0.035$ m, $r_s = 0.07$ m, $r_e = 0.63$ m, $k_w = 2.0 \times 10^{-3}$ cm/s. The soil parameters are listed in Table 3. The coefficient of volume compressibility of the layer 2 is the same as that of the layer 3, but the horizontal coefficient of permeability of natural layer 2, vertical coefficient of permeability of natural layer 2 and horizontal coefficients of permeability of smear layer 2 are 100 times those of layer 3.

Fig. 5 shows excess pore-water pressure isochrones for four different soil profiles at $T = 10$ and $T = 100$. It can be seen that excess pore-water pressure isochrones are dependent on the order of soil layer in the multilayered ground. Change of soil parameters of a soil layer has greater influence on the consolidation of the soil layer below it than that above it. When soil permeability coefficient of a soil layer becomes larger, it will accelerate the consolidation of stratum below the soil layer. Otherwise, it will retard the consolidation of stratum above the soil layer.

The overall average degree of consolidation for the multilayered soil with a vertical drain system under instantaneous loading q_0, defined in terms of excess pore-water pressure, can be given by the following equation:

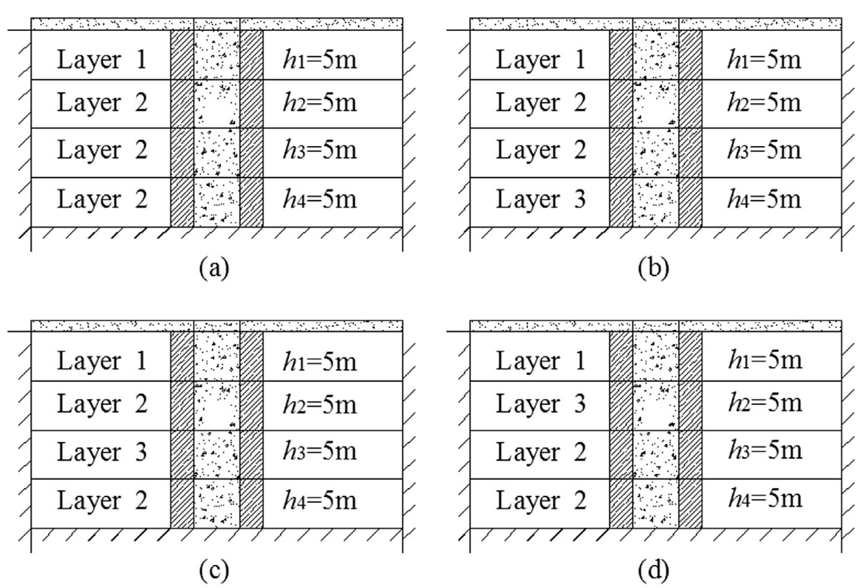

Fig. 4. Assumed soil profiles.

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Table 3
Assumed parameters of different soil layers.

Layer no. i	h_i (m)	k_{vi} (cm/s)	k_{hi} (cm/s)	k_{si} (cm/s)	m_{vi} (kPa^{-1})
1	5.0	1.0×10^{-7}	2.0×10^{-7}	1.0×10^{-7}	1.334×10^{-4}
2	5.0	2.0×10^{-7}	4.0×10^{-7}	2.0×10^{-7}	6.67×10^{-4}
3	5.0	2.0×10^{-5}	4.0×10^{-5}	2.0×10^{-5}	6.67×10^{-4}

$$\overline{U} = h_i \sum_{i=1}^{N} \overline{U_i} \Big/ H_N \tag{28}$$

where $\overline{U_i}$ is the overall average degree of consolidation for each soil layer, which can be expressed as

$$\overline{U_i} = 1 - \frac{1}{h_i} \int_{H_{i-1}}^{H_i} \overline{u_i} \mathrm{d}z \tag{29}$$

where $H_0 = 0$.

Fig. 6 shows the average degrees of consolidation for four different soil profiles developing with time factor T. It can be seen that the order of soil layer in the multilayered ground has great influence on the overall average degree of consolidation. Although the weighted parameters of three soil profiles, namely, case (b), case (c) and case (d), are the same, there is large difference among the average degrees of consolidation. The closer the distance between a soil layer and the pervious top surface, the greater the influence of its consolidation parameter on the overall average degree of consolidation is.

7. Conclusions

Base on the above work, the following conclusions can be obtained:

(1) A general explicit quasi-analytical solution for consolidation of multilayered soil with a vertical drain system is obtained by using the method of Laplace transform and its numerical inverse. Its validity and accuracy are verified by comparing the results obtained from the special cases of the proposed solution with those from a finite-element modeling and an available analytical solution.

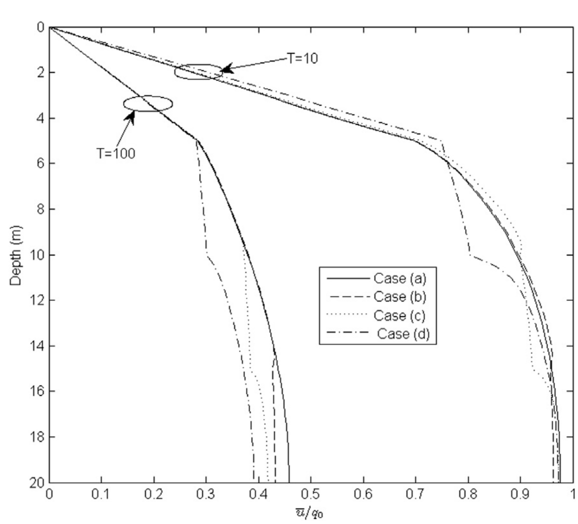

Fig. 5. Excess pore-water pressure distributions for different soil profiles at $T = 10$ and $T = 100$.

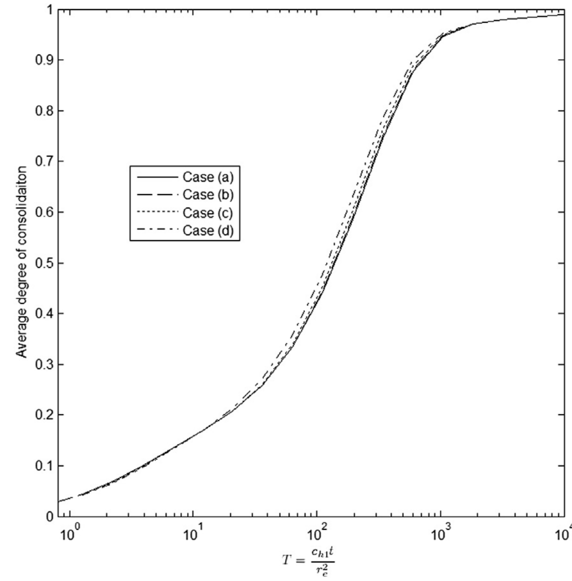

Fig. 6. Average degrees of consolidation for different soil profiles.

(2) The consolidation behavior of a four-layered soil with a vertical drain system is investigated. The order of soil layer in the multilayered ground has a significant effect on the behavior of consolidation. Consolidation rate of a soil layer is mainly controlled by the consolidation parameters of the layer above it. Even if the weighted-average parameters of soil profiles are the same, the excess pore-water pressure isochrones and the average degrees of consolidation may be very different. The closer the distance between a soil layer and the pervious top surface, the greater the influence of its consolidation parameters on the overall average consolidation is. Approximate solution for consolidation of multilayered soil with a vertical drain system should take the order of soil layer in the ground into account.

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Appendix A. The elements of $\mathbf{C}_i$ and $\mathbf{D}_i$ in Eq. (20)

$$\begin{aligned}
[\mathbf{C}_i]_{4\times4} &= \begin{bmatrix} \mathbf{C}_i(1,1) & \mathbf{C}_i(1,2) & \mathbf{C}_i(1,3) & \mathbf{C}_i(1,4) \\ \mathbf{C}_i(2,1) & \mathbf{C}_i(2,2) & \mathbf{C}_i(2,3) & \mathbf{C}_i(2,4) \\ \mathbf{C}_i(3,1) & \mathbf{C}_i(3,1) & \mathbf{C}_i(3,3) & \mathbf{C}_i(3,4) \\ \mathbf{C}_i(4,1) & \mathbf{C}_i(4,2) & \mathbf{C}_i(4,3) & \mathbf{C}_i(4,4) \end{bmatrix} \\
[\mathbf{D}_i]_{4\times4} &= \begin{bmatrix} \mathbf{D}_i(1,1) & \mathbf{D}_i(1,2) & \mathbf{D}_i(1,3) & \mathbf{D}_i(1,4) \\ \mathbf{D}_i(2,1) & \mathbf{D}_i(2,2) & \mathbf{D}_i(2,3) & \mathbf{D}_i(2,4) \\ \mathbf{D}_i(3,1) & \mathbf{D}_i(3,1) & \mathbf{D}_i(3,3) & \mathbf{D}_i(3,4) \\ \mathbf{D}_i(4,1) & \mathbf{D}_i(4,2) & \mathbf{D}_i(4,3) & \mathbf{D}_i(4,4) \end{bmatrix}
\end{aligned} \tag{A1}$$

where $\mathbf{C}_i(1,1) = exp(x_{1i}H_i/H_N)$, $\mathbf{C}_i(1,2) = exp(-x_{1i}H_i/H_N)$, $\mathbf{C}_i(1,3) = exp(x_{2i}H_i/H_N)$, $\mathbf{C}_i(1,4) = exp(-x_{2i}H_i/H_N)$, $\mathbf{C}_i(2,1) = x_{1i}exp(x_{1i}H_i/H_N)$, $\mathbf{C}_i(2,2) = -x_{1i}exp(-x_{1i}H_i/H_N)$, $\mathbf{C}_i(2,3) = x_{2i}exp(x_{2i}H_i/H_N)$, $\mathbf{C}_i(2,4) = -x_{2i}exp(-x_{2i}H_i/H_N)$, $\mathbf{C}_i(3,1) = (1-x_{1i}^2/\varphi_i)\exp(x_{1i}H_i/H_N)$, $\mathbf{C}_i(3,2) = (1-x_{1i}^2/\phi_i)\exp[-x_{1i}(H_i/H_N)]$, $\mathbf{C}_i(3,3) = (1-x_{2i}^2/\varphi_i)\exp \times (x_{2i}(H_i/H_N))$, $\mathbf{C}_i(3,4) = (1-x_{2i}^2/\varphi_i)\exp(-x_{2i}(H_i/H_N))$, $\mathbf{C}_i(4,1) = k_{vi}x_{1i}(1-x_{1i}^2/\varphi_i)\exp(x_{1i}(H_i/H_N))$, $\mathbf{C}_i(4,2) = -k_{vi}x_{1i}(1-x_{1i}^2/\varphi_i)\exp(-x_{1i}(H_i/H_N))$, $\mathbf{C}_i(4,3) = k_{vi}x_{2i}(1-x_{2i}^2/\varphi_i)\exp(x_{2i}(H_i/H_N))$, $\mathbf{C}_i(4,4) = -k_{vi}x_{2i}(1-x_{2i}^2/\varphi_i)\exp(-x_{2i}(H_i/H_N))$, $\mathbf{D}_i(1,1) = exp(x_{1(i+1)}H_i/H_N)$, $\mathbf{D}_i(1,2) = exp(-x_{1(i+1)}H_i/H_N)$, $\mathbf{D}_i(1,3) = exp(x_{2(i+1)}H_i/H_N)$, $\mathbf{D}_i(1,4) = exp(-x_{2(i+1)}H_i/H_N)$, $\mathbf{D}_i(2,1) = x_{1(i+1)}exp(x_{1(i+1)}H_i/H_N)$, $\mathbf{D}_i(2,2) = -x_{1(i+1)}exp(-x_{1(i+1)}H_i/H_N)$, $\mathbf{D}_i(2,3) = x_{2(i+1)}exp(x_{2(i+1)}H_i/H_N)$, $\mathbf{D}_i(2,4) = -x_{2(i+1)}exp(-x_{2(i+1)}H_i/H_N)$, $\mathbf{D}_i(3,1) = (1-x_{1(i+1)}^2/\varphi_{i+1})\exp(x_{1(i+1)}(H_i/H_N))$, $\mathbf{D}_i(3,2) = (1-x_{1(i+1)}^2/\varphi_{i+1})\exp(-x_{1(i+1)}(H_i/H_N))$, $\mathbf{D}_i(3,3) = (1-x_{2(i+1)}^2/\varphi_{i+1})\exp(x_{2(i+1)}(H_i/H_N))$, $\mathbf{D}_i(3,4) = (1-x_{2(i+1)}^2/\varphi_{i+1})\exp(-x_{2(i+1)}(H_i/H_N))$, $\mathbf{D}_i(4,1) = k_{v(i+1)}x_{1(i+1)}(1-x_{1(i+1)}^2/\varphi_{i+1})\exp(x_{1(i+1)}(H_i/H_N))$, $\mathbf{D}_i(4,2) = -k_{v(i+1)}x_{1(i+1)}(1-x_{1(i+1)}^2/\varphi_{i+1})\exp(-x_{1(i+1)}(H_i/H_N))$, $\mathbf{D}_i(4,3) = k_{v(i+1)}x_{2(i+1)}(1-x_{2(i+1)}^2/\varphi_{i+1})\exp(x_{2(i+1)}(H_i/H_N))$, $\mathbf{D}_i(4,4) = -k_{v(i+1)}x_{2(i+1)}(1-x_{2(i+1)}^2/\varphi_{i+1})\exp(-x_{2(i+1)}(H_i/H_N))$.

Appendix B. Derivation of Eq. (27)

Eq. (26) can be written as

$$\tilde{\tilde{u}} = -\tilde{q}\tilde{G}(s) + \tilde{q} \tag{B1}$$

where

$$\begin{aligned}\tilde{G}(s) &= \frac{1}{\exp(2x_1)+1}\frac{x_2^2}{x_2^2-x_1^2}\left(1-\frac{x_1^2}{\varphi}\right)[\exp(x_1 Z)\\ &\quad + \exp(2x_1)\exp(-x_1 Z)] + \frac{1}{\exp(2x_2)+1}\frac{x_1^2}{x_1^2-x_2^2}\\ &\quad \times\left(1-\frac{x_2^2}{\varphi}\right)[\exp(x_2 Z)+\exp(2x_2)\exp(-x_2 Z)]\end{aligned} \tag{B2}$$

It can be derived from Eq. (B2) that $s_k = -M^2c_v r_e^2/(c_h H^2) - \frac{2}{F}M^2/(M^2+\phi)$ (where $M = (2k-1)\pi/2$, $k = 1,2\ldots$) are simple poles of $\tilde{\tilde{u}}$. According to the residue theorem, it can be derived that

$$G(T) = L^{-1}\left[\tilde{G}(s)\right] = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty}\tilde{G}(s)e^{sT}\mathrm{d}s = \sum \mathrm{Res}\left(\tilde{G}(s)e^{sT}, s_k\right) \tag{B3}$$

where Res denotes the residue of $\tilde{G}(s)e^{sT}$ at s_k.

At the simple pole of $s_k = -M^2c_v r_e^2/(c_h H^2) - (2/F)M^2/(M^2+\phi)$, the residue of $\tilde{G}(s)e^{sT}$ is given by

$$\begin{aligned}\mathrm{Res}\left(\tilde{G}(s)e^{sT},\sigma_k\right) &= \lim_{s\to s_k}\left[(s-s_k)\tilde{G}(s)e^{sT}\right]\\ &= e^{s_k T}\lim_{s\to s_k}\left[(s-s_k)\tilde{G}(s)\right]\end{aligned} \tag{B4}$$

By substituting Eq. (B2) into Eq. (B4), the last limit term on the right-hand side of Eq. (B4) can be rewritten as

$$\begin{aligned}\lim_{s\to s_k}\left[(s-s_k)\tilde{G}(s)\right] &= \lim_{s\to s_k}\left\{\frac{s-s_k}{\exp(2x_1)+1}\frac{x_2^2}{x_2^2-x_1^2}\left(1-\frac{x_1^2}{\varphi}\right)\right.\\ &\qquad\left.\times[\exp(x_1Z)+\exp(2x_1)\exp(-x_1Z)]\right\}\\ &\quad+\lim_{s\to s_k}\left\{\frac{s-s_k}{\exp(2x_2)+1}\frac{x_1^2}{x_1^2-x_2^2}\left(1-\frac{x_2^2}{\varphi}\right)\right.\\ &\qquad\left.\times[\exp(x_2Z)+\exp(2x_2)\exp(-x_2Z)]\right\}\end{aligned} \tag{B5}$$

The limits of x_1 and x_2, as s approaches s_k, can be obtained from Eq. (14) as

$$\begin{cases}\lim\limits_{s\to s_k} x_1 = \sqrt{\varphi + \dfrac{2}{F}\dfrac{\varphi}{M^2+\varphi}\dfrac{c_hH^2}{c_v r_e^2}}\\ \lim\limits_{s\to s_k} x_2 = Mj\end{cases} \tag{B6}$$

Substituting Eq. (B6) into Eq. (B5) yields

$$\begin{aligned}\lim_{s\to s_k}\left[(s-s_k)\tilde{G}(s)\right] &= \lim_{s\to s_k}\frac{s-s_k}{\exp(2x_1)+1}\lim_{s\to s_k}\left\{\frac{x_2^2}{x_2^2-x_1^2}\left(1-\frac{x_1^2}{\varphi}\right)[\exp(x_1Z)+\exp(2x_1)\exp(-x_1Z)]\right\}\\ &\quad+\lim_{s\to s_k}\frac{s-s_k}{\exp(2x_2)+1}\lim_{s\to s_k}\left\{\frac{x_1^2}{x_1^2-x_2^2}\left(1-\frac{x_2^2}{\varphi}\right)[\exp(x_2Z)+\exp(2x_2)\exp(-x_2Z)]\right\}\\ &= 0+\frac{\varphi+\frac{2}{F}\frac{\varphi}{M^2+\varphi}\frac{c_hH^2}{c_vr_e^2}}{\varphi+\frac{2}{F}\frac{\varphi}{M^2+\varphi}\frac{c_hH^2}{c_vr_e^2}+M^2}\left(1+\frac{M^2}{\varphi}\right)[2\sin(MZ)j]\lim_{s\to s_k}\frac{s-s_k}{\exp(2x_2)+1}\end{aligned} \tag{B7}$$

By using L'Hôpital's rule, the last limit term on the right-hand side of Eq. (B7) can be rewritten as

$$\begin{aligned}\lim_{s\to s_k}\frac{\frac{d}{ds}(s-s_k)}{\frac{d}{ds}[\exp(2x_2)+1]} &= \lim_{s\to s_k}\left\{\frac{d}{ds}\exp\left[2\sqrt{\frac{\left(s\frac{c_hH^2}{c_vr_e^2}+\frac{c_h}{c_v}\frac{2H^2}{r_e^2F}+\varphi\right)-\sqrt{\left(s\frac{c_hH^2}{c_vr_e^2}+\frac{c_h}{c_v}\frac{2H^2}{r_e^2F}+\varphi\right)^2-4\varphi s\frac{c_h}{c_v}\frac{H^2}{r_e^2}}}{2}}\right]\right\}^{-1} \\ &= \lim_{s\to s_k}\left\{2\exp(2x_2)\times\frac{1}{2x_2}\left[\frac{\frac{c_hH^2}{c_vr_e^2}}{2}-\frac{1}{2}\times\frac{1}{2}\frac{2\left(s\frac{c_hH^2}{c_vr_e^2}+\frac{c_h}{c_v}\frac{2H^2}{r_e^2F}+\varphi\right)\frac{c_hH^2}{c_vr_e^2}-4\varphi\frac{c_h}{c_v}\frac{H^2}{r_e^2}}{\sqrt{\left(s\frac{c_hH^2}{c_vr_e^2}+\frac{c_h}{c_v}\frac{2H^2}{r_e^2F}+\varphi\right)^2-4\varphi s\frac{c_h}{c_v}\frac{H^2}{r_e^2}}}\right]\right\}^{-1} \\ &\quad\times\lim_{s\to s_k}\left\{-2\times\frac{1}{2Mj}\left[\frac{\frac{c_hH^2}{c_vr_e^2}}{2}-\frac{1}{2}\times\frac{1}{2}\frac{2\left(-M^2+\frac{2}{F}\frac{\varphi}{M^2+\varphi}\frac{c_hH^2}{c_vr_e^2}+\varphi\right)\frac{c_hH^2}{c_vr_e^2}-4\varphi\frac{c_h}{c_v}\frac{H^2}{r_e^2}}{\left(M^2+\varphi+\frac{2}{F}\frac{\varphi}{M^2+\varphi}\frac{c_hH^2}{c_vr_e^2}\right)}\right]\right\}^{-1} \\ &= \left\{\frac{-1}{Mj}\left[\frac{M^2\frac{c_hH^2}{c_vr_e^2}+\varphi\frac{c_h}{c_v}\frac{H^2}{r_e^2}}{\left(M^2+\varphi+\frac{2}{F}\frac{\varphi}{M^2+\varphi}\frac{c_hH^2}{c_vr_e^2}\right)}\right]\right\}^{-1} = -Mj\frac{M^2+\varphi+\frac{2}{F}\frac{\varphi}{M^2+\varphi}\frac{c_hH^2}{c_vr_e^2}}{M^2\frac{c_hH^2}{c_vr_e^2}+\varphi\frac{c_h}{c_v}\frac{H^2}{r_e^2}}\end{aligned} \tag{B8}$$

Substituting Eq. (B8) into Eq. (B7) yields

$$\begin{aligned}\lim_{s\to s_k}\left[(s-s_k)\tilde{G}(s)\right] &= \frac{\varphi+\frac{2}{F}\frac{\varphi}{M^2+\varphi}\frac{c_hH^2}{c_vr_e^2}}{\varphi+\frac{2}{F}\frac{\varphi}{M^2+\varphi}\frac{c_hH^2}{c_vr_e^2}+M^2}\left(1+\frac{M^2}{\varphi}\right)[2\sin(MZ)j] \\ &\quad\times\left[-Mj\frac{M^2+\varphi+\frac{2}{F}\frac{\varphi}{M^2+\varphi}\frac{c_hH^2}{c_vr_e^2}}{M^2\frac{c_hH^2}{c_vr_e^2}+\varphi\frac{c_h}{c_v}\frac{H^2}{r_e^2}}\right] \\ &= 2M\left[\frac{c_vr_e^2}{c_hH^2}+\frac{2}{F}\frac{1}{M^2+\varphi}\right]\sin(MZ) = \frac{-2s_k}{M}\sin(MZ)\end{aligned} \tag{B9}$$

$$s_k = -\frac{c_vr_e^2}{c_hH^2}M^2-\frac{2}{F}\frac{M^2}{M^2+\varphi}$$

By substituting Eq. (B9) into Eq. (B4), and then substituting Eq. (B4) into Eq. (B3), $G(T)$ can be derived, as given by

$$G(T) = \sum_{k=1}^{\infty}\frac{-2s_k}{M}\sin(MZ)e^{s_kT} \tag{B10}$$

According to the basic properties of the Laplace transform and the convolution theorem, Eq. (B1) can be written as

$$\begin{aligned}\overline{u} &= L^{-1}\left(\tilde{\overline{u}}\right) = L^{-1}\left[-\tilde{q}(s)\tilde{G}(s)\right]+L^{-1}\left(\tilde{q}\right) \\ &= -\int_0^T q(\tau)G(T-\tau)d\tau+q(T)\end{aligned} \tag{B11}$$

Substituting Eq. (23) into Eq. (B11) yields

$$\begin{aligned}\overline{u} &= \begin{cases}-\sum_{k=1}^{\infty}\int_0^T\frac{q_u\tau}{T_c}\frac{-2s_k}{M}\sin(MZ)e^{s_k(T-\tau)}d\tau+\frac{q_uT}{T_c} & 0\le T\le T_c \\ -\sum_{k=1}^{\infty}\int_0^{T_c}\frac{q_u\tau}{T_c}\frac{-2s_k}{M}\sin(MZ)e^{s_k(T-\tau)}d\tau-\sum_{k=1}^{\infty}\int_{T_c}^{T}q_u\frac{-2s_k}{M}\sin\left(MZ\right)e^{s_k(T-\tau)}d\tau+q_u & T_c\le T\end{cases} \\ &= \begin{cases}\sum_{k=1}^{\infty}\frac{-2}{M}\sin(MZ)\frac{q_u}{T_c}\frac{1}{s_k}\left(1-e^{s_kT}\right) & 0\le T\le T_c \\ \sum_{k=1}^{\infty}\frac{-2}{M}\sin(MZ)\frac{q_u}{T_c}\frac{1}{s_k}\left[e^{s_k(T-T_c)}-e^{s_kT}\right] & T_c\le T\end{cases} = \begin{cases}\frac{q_u}{t_c}\sum_{k=1}^{\infty}\frac{r_e^2}{-s_kc_h}\frac{2}{M}\sin\left(\frac{Mz}{H}\right)\left(1-\exp^{s_kc_ht/r_e^2}\right) & 0\le t\le t_c \\ \frac{q_u}{t_c}\sum_{k=1}^{\infty}\frac{r_e^2}{-s_kc_h}\frac{2}{M}\sin\left(\frac{Mz}{H}\right)\left[\exp^{s_k(t-t_c)c_h/r_e^2}-\exp^{s_ktc_h/r_e^2}\right] & t_c\le t\end{cases}\end{aligned} \tag{B12}$$

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