Journal of the Egyptian Mathematical Society 000 (2016) 1–6

Contents lists available at ScienceDirect

Journal of the Egyptian Mathematical Society

journal homepage: www.elsevier.com/locate/joems

Transient solution of an $M/M/1$ vacation queue with a waiting server and impatient customers

Sherif I. Ammar [a,b]

[a] Mathematics Department, Faculty of Science, Menofia University, Shebin El Kom, Egypt
[b] Department of Mathematics, College of Science, Taibah University, Madinah Munawwarah, Saudi Arabia

ARTICLE INFO

Article history:
Received 13 March 2016
Revised 20 July 2016
Accepted 30 September 2016
Available online xxx

MSC
60K25
68M20
90B22

Keywords:
Transient probabilities
M/M/1 queue
server vacations
A waiting server
Generating functions

ABSTRACT

Recently, Ammar [1] has discussed the transient behavior of a multiple vacations queue with impatient customers. In this paper, a similar technique is used to derive a new elegant explicit solution for an $M/M/1$ vacation queue with impatient customers and a waiting server, where the server is allowed to take a vacation whenever the system is empty after waiting for a random period of time. If the server does not return from the vacation before the expiry of the customer impatience time, the customer abandons the system forever. Moreover, the formulas of mean and variance expressed in terms of the obtained possibilities for this model.

1. Introduction

In recent years, the study of vacation queues has had a great effect on the queueing theory. This has been because of their wide applications in many areas, especially in the computer communication and flexible manufacturing systems. Since Levy and Yechiali [2] have presented a paper about server vacations, many researchers have studied queueing systems with vacations. For the background of such vacations systems (see the excellent overviews of in [3,4] and the monographs in [5,6].

The investigation of queueing models with impatient customers is very helpful and imperative as such systems often arise in many real life problems, see e.g., [7,8]. Therefore, many researchers have studied queueing systems with impatient customers. For related literature, interested readers may refer to [9] and references therein. The studies of queueing systems with impatient customers classified according to the causes of the impatience behavior. Thus, in literatures we accentuation the models that are created by impatient customers due to server vacations.

Recently, Altman and Yechiali [10] have showed a comprehensive analysis of some queueing models such as $M/M/1$, $M/G/1$ and $M/M/c$queue with server vacations and customer impatience, where customers became impatient only when the servers were on vacation. They discussed both single and multiple vacation cases, and obtained various closed-form results. Altman and Yechiali [11] have investigated the infinite server queue with vacations and impatient customers. They have acquired the probability generating function of the number of units in the model and computed values of key performance measures. Perel and Yechiali [12] have studied $M/M/c$queues in a 2-phase (fast and slow) Markovian random environment, with impatient customers. Yue et al. [13] have analyzed an $M/M/1$ queueing system with working vacation and impatient customers. They obtained the probability generating function of the number of units in the model when the server is in a working vacation and a service period. Yue et al. [14] extend the model in [10] by considering a variant of the multiple vacation policy which includes both a single vacation and multiple vacations, they have derived the probability generating functions of the steady state probabilities and obtained the closed form expressions of the system sizes when the server is in different states. Adan et al. [15] have addressed queueing systems with vacations and synchronized reneging.

Padmavathy et al. [16] have studied the steady state behavior of vacation queues with impatient customers and a waiting Server.

E-mail address: sherifamar2000@yahoo.com

http://dx.doi.org/10.1016/j.joems.2016.09.002
1110-256X/

In spite of the great interest in studying of queuing systems with vacations, a few works have dealt with the transient solution of these systems. Kalidass et al. [17] have discussed the transient behavior of an $M/M/1$ multiple vacation queue and the possibilities of catastrophes. Sudhesh and Francis Raj [18] have obtained the time dependent system size probabilities of a $M/M/1$ queue with working vacation. Indra and Sweety [19] have derived the transient solution of an unreliable $M/G/1$ vacations queue.

Recently, Ammar [1] has investigated the transient solution of a $M/M/1$ multiple vacations queue and impatient customers.

As we mentioned earlier in all the previous studies, the server leaves the system just as the system is empty of customers, but in the practical life the server waits a certain period of time even if the system is empty, especially, if we are dealing with human behavior. This conduct of the server was initially presented in [20]. Based on [20] Yechiali [21] has extended the analysis to batch arrival queue.

Therefore, the main aim of this article is to study the $M/M/1$ vacation queue with impatient customers and a waiting server. We have obtained in closed form the transient probabilities, mean and variance. This work may be regarded as an extension to work in [22] where the transient probabilities of the system are derived without impatient customers.

2. System model

We consider a $M/M/1$ vacation queue with customer's impatience and a waiting server. The assumption of the model are as follows:

a) Customers arriving according to a Poisson process with rate λ and the server has an independently and identically distributed exponential service time distribution with mean service discipline is FCFS and there is infinite room for customers to wait.
b) When the busy period is ended the server waits a random duration of time before beginning on a vacation. This waiting duration follows the exponentially distributed with the density function as follows:

$$w(t) = \eta e^{-\eta t} \qquad t \geq 0, \quad \eta \geq 0$$

where η is the waiting rate of a server.
c) It is assumed that the interval of vacation has an exponential distribution with the density function as follows:

$$v(t) = \gamma e^{-\gamma t} \qquad t \geq 0, \quad \gamma \geq 0$$

where γ is the vacation rate of a server.
d) When the server is on a vacation, each customer sets up an impatience timer independently of the other customers in the system, which is assumed to be exponentially distributed with the density function as follows:

$$s(t) = \xi e^{-\xi t} \qquad t \geq 0, \quad \xi \geq 0$$

where ξ is the impatience rate of a server.
e) If the impatience timer expires while the server is on a vacation, the customer abandons the queue, never to return.

3. Transient behavior

Let $N(t)$be the number of units in the system at time t, and $X(t)$ denote the system state at time$t$. If $X(t)=1$, the server is working and serving units, whilst if $X(t)=0$, the server is on vacation. Then $\{X(t),\ N(t),\ t \geq 0\}$is a continuous time Markov chain. Let $P_{ij}(t)=P[X(t)=i,\ N(t)=j]$ denote the system state in the transient probabilities. These probabilities satisfy the forward Kolmogorov differential-difference equations are given by:

$$P'_{00}(t) = -(\lambda+\gamma)P_{00}(t) + \xi P_{01}(t) + \eta P_{10}(t) \tag{3.1}$$

$$P'_{0n}(t) = \lambda P_{0,n-1}(t) - (\lambda + n\xi + \gamma)P_{0n}(t) + (n+1)\xi P_{0,n+1}(t), \quad n = 1, 2, 3, \ldots \tag{3.2}$$

$$P'_{10}(t) = -(\lambda+\eta)P_{10}(t) + \mu P_{11}(t) + \gamma P_{00}(t) \tag{3.3}$$

$$P'_{1n}(t) = \lambda P_{1,n-1}(t) - (\lambda+\mu)P_{1n}(t) + \mu P_{1,n+1}(t) + \gamma P_{0n}(t), \quad n = 1, 2, 3, \ldots \tag{3.4}$$

and suppose that initially there is no unit in the system.

3.1. Time dependent probabilities

3.1.1. Evaluation for $P_{0n}(t)$

We obtain expression for $P_{0n}(t)$by employing the continued fraction and well-known identities of confluent hypergeometric function. In the sequel, $\hat{g}(s)$denotes the Laplace transform of $g(t)$..

Now, by taking the Laplace transforms on (3.2), we get.

$$\frac{\hat{P}_{0n}(s)}{\hat{P}_{0,n-1}(s)} = \frac{\lambda}{s+\lambda+\gamma+n\xi-(n+1)\xi\frac{\hat{P}_{0,n+1}(s)}{\hat{P}_{0n}(s)}}$$

The above equation can be written as a continued fraction as follows,

$$\frac{\hat{P}_{0n}(s)}{\hat{P}_{0,n-1}(s)} = \frac{\lambda}{s+\lambda+\gamma+n\xi-\frac{(n+1)\xi\lambda}{s+\lambda+\gamma+(n+1)\xi-\frac{(n+2)\xi\lambda}{s+\lambda+\gamma+(n+2)\xi-\frac{(n+3)\xi\lambda}{s+\lambda+\gamma+(n+3)\xi-\ldots}}}} \tag{3.5}$$

By means of the properties of confluent hypergeometric function. The Eq. (3.5) will take the following form

$$\frac{\hat{P}_{0n}(s)}{\hat{P}_{0,n-1}(s)} = \frac{\lambda}{\xi}\frac{{}_1F_1(n+1;\ \frac{s+\gamma}{\xi}+n+1;\ \frac{-\lambda}{\xi})}{(\frac{s+\gamma}{\xi}+n)F_1(n;\ \frac{s+\gamma}{\xi}+n;\ \frac{-\lambda}{\xi})} \tag{3.6}$$

Invoking of the above equation we can obtain for $n=1,\ 2,\ 3,\ \ldots$

$$\hat{P}_{0n}(s) = \left(\frac{\lambda}{\xi}\right)^n \frac{1}{\prod_{j=1}^{n}(\frac{s+\gamma}{\xi}+j)}\frac{{}_1F_1(n+1;\ \frac{s+\gamma}{\xi}+n+1;\ \frac{-\lambda}{\xi})}{(\frac{s+\gamma}{\xi}+n){}_1F_1(1;\ \frac{s+\gamma}{\xi}+1;\ \frac{-\lambda}{\xi})}P_{00}(s) = \hat{\Phi}_n(s)\hat{P}_{00}(s)$$

Then

$$P_{0n}(s) = \Phi_n(t) * P_{00}(t) \tag{3.7}$$

where $\Phi_n(t)$ is the inverse transform of $\hat{\Phi}_n(s)$ and the formula of it given in Section 3.7, and where * denotes convolution.

3.1.2. Evaluation for $P_{1n}(t)$

We will evaluate the probability $P_{1n}(t), \quad n \geq 1$.

Define

$$P_i(z,t) = \sum_{n=1}^{\infty} P_{in}(t)z^n \quad i = 0,\ 1. \tag{3.8}$$

The system of Eqs. (3.3) and (3.4) yield

$$\frac{\partial P_1(z,t)}{\partial t} = \left[\lambda(z-1) - \mu\left(1-\frac{1}{z}\right)\right]P_1(z,t) + \left[\mu\left(1-\frac{1}{z}\right) - \eta\right] \times P_{10}(t) + \gamma P_0(z,t)$$

Integrating

$$P_1(z,t) = e^{-[(\lambda z+\frac{\mu}{z})-(\lambda+\mu)]t} + \left[(\mu-\eta)-\frac{\mu}{z})\right] \times \int_0^t P_{10}(v)e^{-((\lambda z+\frac{\mu}{z})-(\lambda+\mu))(t-v)}dv + \gamma\int_0^t P_0(z,v)e^{-((\lambda z+\frac{\mu}{z})-(\lambda+\mu))(t-v)}dv \quad (3.9)$$

It is well known that if $\alpha = 2\sqrt{\lambda\mu}$ and $\beta = \sqrt{\lambda/\mu}$, then

$$\exp\left[\left(\lambda z+\frac{\mu}{z}\right)t\right]\alpha = \sum_{n=-\infty}^{\infty}(\beta z)^n I_n(\alpha t),$$

where $I_n(.)$ is the modified Bessel function.

Comparing the coefficients of z^n on right and left hand sides in (3.9), we get for $n=1, 2, 3,\ldots$

$$P_{1n}(t) = e^{-(\lambda+\mu)t}\beta^n I_n(\alpha t) + (\mu-\eta)\beta^n \times \int_0^t P_{10}(v)I_n(\alpha(t-v))e^{-(\lambda+\mu)(t-v)}dv - \mu\beta^{n+1}\int_0^t P_{10}(v)I_{n+1}(\alpha(t-v))e^{-(\lambda+\mu)(t-v)}dv + \gamma\int_0^t\sum_{k=0}^{\infty}P_{0k}(v)\beta^{n-k}I_{n-k}(\alpha(t-v))e^{-(\lambda+\mu)(t-v)}dvdv \quad (3.10)$$

Taking the Laplace transform of the above equations and simplifying, we have

$$\hat{P}_{1n}(s) = \frac{1}{\sqrt{p^2-\alpha^2}}\left(\frac{p-\sqrt{p^2-\alpha^2}}{2\mu}\right)^n + \frac{(\mu-\eta)}{\sqrt{p^2-\alpha^2}}\left(\frac{p-\sqrt{p^2-\alpha^2}}{2\mu}\right)^n\hat{P}_{10}(s) - \frac{\mu}{\sqrt{p^2-\alpha^2}}\left(\frac{p-\sqrt{p^2-\alpha^2}}{2\mu}\right)^{n+1}\hat{P}_{10}(s) + \frac{\gamma}{\sqrt{p^2-\alpha^2}}\sum_{k=0}^{n-1}\hat{P}_{0k}(s)\left(\frac{p-\sqrt{p^2-\alpha^2}}{2\mu}\right)^{n-k} + \frac{\gamma\,\Phi_n(s)}{\sqrt{p^2-\alpha^2}}\sum_{r=0}^{\infty}\hat{P}_{0r}(s)\left(\frac{p-\sqrt{p^2-\alpha^2}}{2\mu}\right)^r \quad (3.11)$$

where $p=s+\lambda+\mu$

As $P_i(z, t)$ does not contain terms with negative powers of z the right-hand side of (3.10) with $n$replaced by $-n$ must be zero. Thus,

$$0 = e^{-(\lambda+\mu)t}\beta^n I_n(\alpha t) + (\mu-\eta)\beta^{-n} \times \int_0^t P_{10}(v)I_n(\alpha(t-v))e^{-(\lambda+\mu)(t-v)}dv - \mu\beta^{-n+1}\int_0^t P_{10}(v)I_{n+1}(\alpha(t-v))e^{-(\lambda+\mu)(t-v)}dv + \gamma\int_0^t\sum_{k=0}^{\infty}P_{0k}(v)\beta^{-(n+k)}I_{n+k}(\alpha(t-v))e^{-(\lambda+\mu)(t-v)}dvdv \quad (3.12)$$

using $I_{-n}(x)=I_n(x)$.

For $n=0$,

$$0 = e^{-(\lambda+\mu)t}\beta^n I_0(\alpha t) + (\mu-\eta)\beta^{-n} \times \int_0^t P_{10}(v)I_0(\alpha(t-v))e^{-(\lambda+\mu)(t-v)}dv - \mu\beta^{-n+1}\int_0^t P_{10}(v)I_1(\alpha(t-v))e^{-(\lambda+\mu)(t-v)}dv + \gamma\int_0^t\sum_{k=0}^{\infty}P_{0k}(v)\beta^{-k}I_k(\alpha(t-v))e^{-(\lambda+\mu)(t-v)}dvdv \quad (3.13)$$

The Laplace transform of Eq. (3.13) given as follows

$$\frac{\gamma}{\sqrt{p^2-\alpha^2}}\sum_{r=0}^{\infty}\hat{P}_{0r}(s)\left(\frac{p-\sqrt{p^2-\alpha^2}}{2\mu}\right)^r = \frac{-1}{\sqrt{p^2-\alpha^2}}\left\{1+(\mu-\eta)\hat{P}_{10}(s)-\sqrt{p^2-\alpha^2}\hat{P}_{10}(s) - \frac{1}{2}(p-\sqrt{p^2-\alpha^2})\hat{P}_{10}(s)\right\} \quad (3.14)$$

Substituting (3.14) in (3.11), and considerably simplifying the working, we obtain

$$\hat{P}_{1n}(s) = \frac{\beta^n}{\sqrt{p^2-\alpha^2}}\left(\frac{p-\sqrt{p^2-\alpha^2}}{2\mu}\right)^n + \frac{(\mu-\eta)\beta^n}{\sqrt{p^2-\alpha^2}}\left(\frac{p-\sqrt{p^2-\alpha^2}}{2\mu}\right)^n\hat{P}_{10}(s) - \frac{\mu\beta^{n+1}}{\sqrt{p^2-\alpha^2}}\left(\frac{p-\sqrt{p^2-\alpha^2}}{2\mu}\right)^{n+1}\hat{P}_{10}(s) + \frac{\gamma\beta^{n-k}}{\sqrt{p^2-\alpha^2}}\sum_{k=0}^{n-1}\hat{P}_{0k}(s)\left(\frac{p-\sqrt{p^2-\alpha^2}}{2\mu}\right)^{n-k} + \frac{\Phi_n(s)}{\sqrt{p^2-\alpha^2}}\left\{1+(\mu-\eta)\hat{P}_{10}(s)-\sqrt{p^2-\alpha^2}\hat{P}_{10}(s) - \frac{1}{2}\left(p-\sqrt{p^2-\alpha^2}\right)\hat{P}_{10}(s)\right\}$$

which on inversion yields an explicit expression for us the following equation $P_{1n}(t)$ given by

$$P_{1n}(t) = \beta^n(I_{n-1}(\alpha t)-I_{n+1}(\alpha t))e^{-(\lambda+\mu)t} + (\mu-\eta)\beta^{-n}P_{10}(t)*I_n(\alpha t)e^{-(\lambda+\mu)t} - \mu\beta^{-n+1}P_{10}(t)*I_{n+1}(\alpha t)e^{-(\lambda+\mu)t} + \Phi_n(t)*I_0(\alpha t)e^{-(\lambda+\mu)t} - (\mu-\eta)P_{10}(t)*\Phi_n(t)*I_0(\alpha t)e^{-(\lambda+\mu)t} + \Phi_n(t)*P_{10}(t) + \frac{\alpha}{2}\Phi_n(t)*P_{10}(t)*I_1(\alpha t)e^{-(\lambda+\mu)t} + \gamma\sum_{k=0}^{n-1}\beta^{n-k}P_{0k}(t)I_{n-k}(\alpha t)\,e^{-(\lambda+\mu)t} \quad (3.15)$$

Thus we have expressed $P_{1n}(t)$ in terms of $P_{0n}(t)$ and $P_{10}(t)$.

3.1.3. Evaluation for $P_{10}(t)$

Comparing the coefficients of z^{-1} on both sides of Eq. (3.12) and using $I_{-n}(x)=I_n(x)$, we get

$$0 = e^{-(\lambda+\mu)t}\beta^{-1}I_1(\alpha t) + (\mu-\eta)\beta^{-1} \times \int_0^t P_{10}(v)I_1(\alpha(t-v))e^{-(\lambda+\mu)(t-v)}dv$$

$$-\mu\int_0^t P_{10}(v)I_0(\alpha(t-v))e^{-(\lambda+\mu)(t-v)}dv + \gamma\int_0^t\sum_{k=0}^{\infty}P_{0k}(v)\beta^{-(k+1)}I_{k+1}(\alpha(t-v))e^{-(\lambda+\mu)(t-v)}dvdv \tag{3.16}$$

Taking Laplace transform of (3.16) and simplifying, we get

$$\hat{P}_{10}(s) = \frac{1}{2\lambda\mu}\sum_{r=0}^{\infty}(-1)^{r+1}\left(\frac{\mu-\eta}{2\lambda\mu}\right)^r\left(p-\sqrt{p^2-\alpha^2}\right)^{r+1} + \frac{\gamma}{2\lambda\mu}\sum_{r=0}^{\infty}(-1)^{r+1}\left(\frac{\mu-\eta}{2\lambda\mu}\right)^r\left(p-\sqrt{p^2-\alpha^2}\right)^{r+1} \times\sum_{k=0}^{\infty}\hat{P}_{0k}(s)\left(\frac{p-\sqrt{p^2-\alpha^2}}{\alpha\beta}\right)^k \tag{3.17}$$

Inverting (3.17) we obtain the expression of $P_{10}(t)$ as follows

$$P_{10}(t) = \frac{1}{2\lambda\mu}\sum_{r=0}^{\infty}(-1)^{r+1}\left(\frac{\alpha(\mu-\eta)}{2\lambda\mu}\right)^r[I_r(\alpha t)-I_{r+2}(\alpha t)]e^{-(\lambda+\mu)t} + \frac{\alpha\gamma}{2}\sum_{r=0}^{\infty}(-1)^{r+1}\left(\frac{\alpha(\mu-\eta)}{2\lambda\mu}\right)^r \times\left[(I_r(\alpha t)-I_{r+2}(\alpha t))e^{-(\lambda+\mu)t}\right] *\sum_{k=0}^{\infty}\beta^{-k}P_{0k}(t)*[I_{k-1}(\alpha t)-I_{k+1}(\alpha t)]e^{-(\lambda+\mu)t} \tag{3.18}$$

3.1.4. Evaluation for $P_{00}(t)$

On applying Laplace transform to the system Eq. (3.1), we have

$$(s+\lambda+\gamma)P_{00}(s) = \xi P_{01}(s)+\eta P_{10}(s) \tag{3.19}$$

Now, using (3.7) and (3.17) in (3.19) and after some mathematical manipulation, we obtain

$$\hat{P}_{00}(s) = \frac{\eta}{2\lambda\mu}\sum_{m=0}^{\infty}\sum_{j=0}^{m}(-1)^m\binom{m}{j}\left(\frac{\gamma}{2\lambda\mu}\right)^{m-j} \times\xi^j\left(\frac{1}{s+\lambda+\gamma}\right)^{m+1}\hat{\Phi}_1^r(s) \times\left[\sum_{r=0}^{\infty}(-1)^{r+1}\left(\frac{\mu-\eta}{2\lambda\mu}\right)^r\left(p-\sqrt{p^2-\alpha^2}\right)^{r+1}\right]^{m-j+1} \times\left[\sum_{r=0}^{\infty}\hat{\Phi}_k(s)\left(\frac{p-\sqrt{p^2-\alpha^2}}{\alpha\beta}\right)^k\right]^{m-j}$$

which on inversion yields an explicit expression for $P_{00}(t)$ as

$$P_{00}(t) = \frac{\eta}{2\lambda\mu}\sum_{m=0}^{\infty}\sum_{j=0}^{m}(-1)^m\binom{m}{j}\left(\frac{\gamma}{2\lambda\mu}\right)^{m-j}\xi^j e^{-(\lambda+\gamma)t}\frac{t^m}{m!}*\Phi_1^r(t) *\left[\sum_{r=0}^{\infty}(-1)^{r+1}\left(\frac{\alpha(\mu-\eta)}{2\lambda\mu}\right)^r(I_r(\alpha t)-I_{r+2}(\alpha t))e^{-(\lambda+\gamma)t}\right]^{*(m-j+1)} \times\left[\sum_{r=0}^{\infty}\beta^{j-m}[I_{k-1}(\alpha t)-I_{k+1}(\alpha t)]e^{-(\lambda+\gamma)t}*\Phi_k(t)\right]^{*(m-j)} \tag{3.20}$$

where $*(m-j+1)$ denotes $(m-j+1)$-fold convolution and $*(m-j)$ denotes $(m-j)$-fold convolution.

Thus, Eqs. (3.7), (3.15), (3.18) and (3.20) taken together complete the transient solution.

3.2. Performance measures

3.2.1. Mean

Let $E(V(t))$ be the average number of customers in the model at time t, then $E(V(t))$ is given by the expression

$$E(V(t)) = m(t) = \sum_{n=1}^{\infty}n(P_{0n}(t)+P_{1n}(t))$$

$$m'(t) = \sum_{n=1}^{\infty}n(P'_{0n}(t)+P'_{1n}(t))$$

From Eqs. (3.2)–(3.4) and after considerable mathematical manipulations, the above equation will lead to the following differential equation

$$m'(t) = \lambda-\mu+\mu P_{10}(t)+\mu\sum_{n=1}^{\infty}P_{0n}(t)-\xi\sum_{n=1}^{\infty}nP_{0n}(t)$$

Therefore,

$$m(t) = (\lambda-\mu)t+\mu\int_0^t P_{10}(y)\,dy + \mu\sum_{n=1}^{\infty}\int_0^t P_{0n}(y)dy-\xi\sum_{n=1}^{\infty}n\int_0^t P_{0n}(y)dy \tag{3.21}$$

where $P_{0n}(t)$ and $P_{10}(t)$ are given in (3.7) and (3.18).

3.2.2. Variance

Let $Var(V(t))$ be the variance number of customers in the model at time t, then $Var(V(t))$ is given by the expression

$$Var(V(t)) = E[V^2(t)]-[E(V(t))]^2$$
$$Var(V(t)) = u(t)-[m(t)]^2$$

where

$$u(t) = E[V^2(t)] = \sum_{n=1}^{\infty}n^2(P'_{0n}(t)+P'_{1n}(t))$$

From Eqs. (3.2)–(3.4) and after considerable mathematical manipulations, the above equation will lead to the following differential equation

$$u'(t) = (\lambda+\mu)+2(\lambda-\mu)m(t)-2\xi\sum_{n=1}^{\infty}n^2P_{0n}(t) + \xi\sum_{n=1}^{\infty}nP_{0n}(t)+2\mu\sum_{n=1}^{\infty}nP_{0n}(t)-\mu\sum_{n=1}^{\infty}P_{0n}(t)-\mu P_{10}(t)$$

Therefore,

$$Var(v(t)) = (\lambda+\mu)t+2(\lambda-\mu)\int_0^t m(y)dy -2\xi\sum_{n=1}^{\infty}n^2\int_0^t P_{0n}(y)dy+\xi\sum_{n=1}^{\infty}n\int_0^t P_{0n}(y)dy +2\mu\sum_{n=1}^{\infty}n\int_0^t P_{0n}(y)dy-\mu\sum_{n=1}^{\infty}\int_0^t P_{0n}(y)dy -\mu\int_0^t P_{0n}(y)dy-[m(t)]^2 \tag{3.22}$$

3.3. Expression for $\Phi_n(t)$

$$\Phi_n(s) = \left(\frac{\lambda}{\xi}\right)^n\frac{1}{\prod_{j=1}^{n}(\frac{s+\gamma}{\xi}+j)}\frac{{}_1F_1(n+1;\ \frac{s+\gamma}{\xi}+n+1;\ \frac{-\lambda}{\xi})}{(\frac{s+\gamma}{\xi}+n){}_1F_1(1;\ \frac{s+\gamma}{\xi}+1;\ \frac{-\lambda}{\xi})} \tag{3.7.1}$$

The Eq. (3.7.1) takes the form

$$\frac{{}_1F_1(n+1;\ \frac{s+\gamma}{\xi}+n+1;\ \frac{-\lambda}{\xi})}{\prod_{j=1}^{n}(\frac{s+\gamma}{\xi}+j)} = \xi^n \sum_{m=0}^{\infty} \frac{\binom{n+m}{m}(-\lambda)^m}{\prod_{j=1}^{n+m}(s+\gamma+j\xi)} \quad (3.7.2)$$

By resolving into partial fractions, we have

$$\frac{{}_1F_1(n+1;\ \frac{s+\gamma}{\xi}+n+1;\ \frac{-\lambda}{\xi})}{\prod_{j=1}^{n}(\frac{s+\gamma}{\xi}+j)} = \xi^n \sum_{m=0}^{\infty} \binom{n+m}{m}\left(\frac{-\lambda}{\xi}\right) \times \sum_{j=1}^{n+m} \frac{(-1)^{j-1}}{(j-1)!(n+m-j)!(s+\gamma+j\xi)} \quad (3.7.3)$$

Also,

$${}_1F_1\left(1;\ \frac{s+\gamma}{\xi}+1;\ \frac{-\lambda}{\xi}\right) = \sum_{m=0}^{\infty} \frac{(-\lambda)^m}{\prod_{j=1}^{m}(s+\gamma+j\xi)} = \sum_{m=0}^{\infty}(-\lambda)^m d_m(s), \quad m=1,\ 2,\ 3, \ldots$$

where,

$$d_m(s) = \sum_{m=0}^{\infty} \frac{(-\lambda)^m}{\prod_{j=1}^{m}(s+\gamma+j\xi)} = \sum_{r=1}^{m} \frac{(-1)^{r-1}}{(r-1)!(m-r)!(s+\gamma+r\xi)} \quad m=1,\ 2,\ 3, \ldots$$

Using the equality given in [23], we get

$$\left[{}_1F_1\left(1;\ \frac{s+\gamma}{\xi}+1;\ \frac{-\lambda}{\xi}\right)\right]^{-1} = \sum_{m=0}^{\infty} g_m(s)\lambda^m, \quad (3.7.4)$$

where $g_0(s)=1$ and for $m=1,\ 2,\ 3,\ldots$

$$g_m(s) = \begin{vmatrix} d_1(s) & 1 & & \cdots & & \\ d_2(s) & d_1(s) & 1 & \cdots & & \\ d_3(s) & d_2(s) & d_1(s) & \cdots & & \\ \vdots & \vdots & \vdots & & & \\ d_{m-1}(s) & d_{m-2}(s) & d_{m-3}(s) & \cdots & d_1(s) & 1 \\ d_m(s) & d_{m-1}(s) & d_{m-2}(s) & \cdots & d_2(s) & d_1(s) \end{vmatrix} = \sum_{j=1}^{m}(-1)^{j-1} d_j(s) g_{m-j}(s).$$

By substituting (3.7.3) and (3.7.4) in (3.7.1), we obtain,

$$\Phi_n(s) = \lambda^n \sum_{m=0}^{\infty} \lambda^{-k} \binom{n+k}{k}\left(\frac{-\lambda}{\xi}\right) d_{n+k}(s) \sum_{m=1}^{\infty} g_m(s)\lambda^m$$

On inversion,

$$\Phi_n(t) = \lambda^n \sum_{m=0}^{\infty} \lambda^{-k} \binom{n+k}{k}\left(\frac{-\lambda}{\xi}\right) d_{n+k}(t) * \sum_{m=1}^{\infty} g_m(t)\lambda^m$$

where

$$d_m(t) = \sum_{r=1}^{m} \frac{(-1)^{r-1}}{(r-1)!(m-r)!} e^{-(\gamma+r\xi)t}, \quad m=1,\ 2,\ 3, \ldots$$

$$g_m(t) = \sum_{j=1}^{m}(-1)^{j-1} d_j(t) * g_{m-j}(t), \quad m=2,\ 3,\ 4,\ \ldots;$$

$$d_1(t) = g_1(t)$$

4. Conclusion

In this paper, we discussed the transient solution of an $M/M/1$ queue with impatient customers and server vacations under a waiting server. We have derived closed form explicit expressions analytically for the system size probabilities, mean and variance by employing Laplace transforms, continued fractions and generating functions. These expressions can be easily evaluate numerically if desired.

A. Appendix: confluent hypergeometric function

We display the definition of confluent hypergeometric function and some properties of this function. The confluent hypergeometric function is denoted by ${}_1F_1(a;\ c;\ z)$ and is defined by the power series

$${}_1F_1(a;c;z) = 1 + \frac{a}{c}\frac{z}{1!} + \frac{a}{c}\frac{(a+1)}{(c+1)}\frac{z^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k}\frac{z^k}{k!} \quad (2.1)$$

provided that c does not equal 0, −1, −2,.... Here $(\alpha)_k$ is the rising factorial function (the Pochhammer symbol), which is defined by:

$$(\alpha)_n = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}$$

We observe that

$${}_1F_1(0;c;z) = 1.$$

The quotient of two hypergeometric functions may be expressed as continued fractions. The following identity from Lorentzen and Waadeland [24].

$$\frac{{}_1F_1(a+1;c+1;z)}{{}_1F_1(a;c;z)} = \frac{c}{c-z+}\,\frac{(a+1)z}{c-z+1+}\,\frac{(a+2)z}{c-z+2+}\cdots,$$

which can be rewritten as

$$c\frac{{}_1F_1(a;c;z)}{{}_1F_1(a+1;c+1;z)} - (c-z) = \frac{(a+1)z}{c-z+1+}\,\frac{(a+2)z}{c-z+2+}\cdots. \quad (2.2)$$

References

[1] S.I. Ammar, Transient analysis of an $M/M/1$ queue with impatient behavior and multiple vacations, Appl. Math. Comput. 260 (2015) 97–105.
[2] Y. Levy, U. Yechiali, Utilization of idle time in an $M/G/1$ queueing system, Manage. Sci. 22 (1975) 202–211.
[3] B. Doshi, Queueing systems with vacations - a survey, Queueing Syst. 1 (1986) 29–66.
[4] H. Takagi, in: Queueing Analysis: A Foundation of Performance Evaluation, North-Holland, Amsterdam, 1991, p. 1.
[5] N. Tian, Z. Zhang, Vacation Queueing Models-Theory and Applications, Springer-Verlag, New York, 2006.
[6] J.C. Ke, C.H. Wu, Z.G. Zhang, Recent development in vacation queueing models: a short survey, Int. J. Oper. Res. 7 (2010) 3–8.
[7] S. Benjaafar, J. Gayon, S. Tepe, Optimal control of a production-inventory system with customer impatience, Oper. Res. Lett. 38 (2010) 267–272.
[8] T. Bonald, J. Roberts, Performance modeling of elastic traffic in overload, in: ACM Sigmetrics Performance Evaluation Review, 29, 2001, pp. 342–343.
[9] C.H. Wu, J.C. Ke, Computational algorithm and parameter optimization for a multi-server system with unreliable servers and impatient customers, J. Comput. Appl. Math. 235 (2010) 547–562.
[10] E. Altman, U. Yechiali, Analysis of customers' impatience in queues with server vacation, Queueing Syst. 52 (2006) 261–279.
[11] E. Altman, U. Yechiali, Infinite server queues with systems' additional task and impatient customers, Probab. Eng. Inf. Sci. 22 (2008) 477–493.
[12] N. Perel, U. Yechiali, Queues with slow servers and impatient customers, Eur. J. Oper. Res. 201 (2010) 247–258.
[13] D. Yue, W. Yue, G. XU, Analysis of customers impatience in an M/M/1 queue with working vacations, J. Ind. Manag. Optim. 8 (2012) 895–908.
[14] D. Yue, W. Yue, Z. Saffer, X. Chen, Analysis of an M/M/1 queueing system with impatient customers and a variant of multiple vacation policy, in: The 7th International Conference on Queueing Theory and Network Applications (QTNA2012), Kyoto, Japan, August 2012, pp. 1–3.
[15] I. Adan, A. Economou, S. Kapodistria, Synchronized reneging in queueing systems with vacations, Queueing Syst. 62 (2009) 1–33.
[16] R. Padmavathy, K. Kalidass, K. Ramanath, Vacation queues with impatient customers and a waiting server, Int. J. Latest Trends Softw. Eng. 1 (2011) 10–19.
[17] K. Kalidass, J. Gnanaraj, S. Gopinath, K. Ramanath, Transient analysis of an M/M/1 queue with a repairable server and multiple vacations, Int. J. Math. Oper. Res. 6 (2014) 193–216.

Please cite this article as: S.I. Ammar, Transient solution of an M/M/1 vacation queue with a waiting server and impatient customers, Journal of the Egyptian Mathematical Society (2016), http://dx.doi.org/10.1016/j.joems.2016.09.002

[18] R. Sudhesh, L. Francis Raj, Computational analysis of stationary and transient distribution of single server queue with working vacation, in: Global Trends in Computing and Communication Systems, 269, 2012, pp. 480–489.
[19] Indra, B Sweety, The transient solution of an unreliable M/G/1 queue with vacations, Int. J. Inf. Manage. Sci. 21 (2010) 391–406.
[20] J. Boxma, S. Schlegel, U. Yechiali, A note on an M/G/1 queue with a waiting server timer and vactions, Am. Math. Soc. Translations 207 (2002) 25–35 Series 2.
[21] U. Yechiali, On the M^x/G/1 queue with a waiting server and vacations, Sankhya 66 (2004) 1–17.
[22] K. Kalidass, K. Ramanath, Time dependent analysis of M/M/1 queue with server vacations and a waiting server, in: The 6th International Conference on Queueing Theory and Network Applications (QTNA2011), Seoul, Korea, 2011, pp. 23–26.
[23] I Gradshteyn, A.J. Ryzhik, D. Zwillinger, Table of Integrals, Series, and Products, seventh ed., Academic Press is an imprint of Elsevier, 2007.
[24] L. Lorentzen, H. Waadeland, Continued fractions with applications, Studies in Computational Mathematics, Elsevier, Amsterdam, 1992 3.

Please cite this article as: S.I. Ammar, Transient solution of an M/M/1 vacation queue with a waiting server and impatient customers, Journal of the Egyptian Mathematical Society (2016), http://dx.doi.org/10.1016/j.joems.2016.09.002