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Non-local finite element analysis of damped beams

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Abstract

Non-local viscoelastic beam models are used to analyse the dynamics of beams with different boundary conditions using the finite element method. Unlike local damping models the internal force of the non-local model is obtained as weighted average of state variables over a spatial domain via convolution integrals with spatial kernel functions that depend on a distance measure. In the finite element analysis, the interpolating shape functions of the element displacement field are identical to those of standard two-node beam elements. However, for non-local damping, nodes remote from the element do have an effect on the energy expressions, and hence on the damping matrix. The expressions of these direct and cross damping matrices may be obtained explicitly for some common spatial kernel functions and Euler–Bernoulli beam theory. Alternatively numerical integration may be applied to obtain solutions. Examples are given where the eigenvalues are compared to the exact solution for a pinned–pinned beam to demonstrate the convergence of the finite element method. The results for beams with other boundary conditions are used to demonstrate the versatility of the finite element technique.

Keywords: Non-local damping; Finite element analysis; Beams

1. Introduction

The finite element method, coupled with model updating techniques, enables the mass and the stiffness properties of complex engineering structures to be accurately modelled, and their dynamics analysed. The design of a structure also requires the determination of the dynamic response, and this also depends on the energy dissipation properties or *damping*. The modelling capability of damping properties, and the analysis procedures for the resulting models, are not as advanced as those based on just the mass and stiffness properties. There are several reasons for this, including: (a) in contrast to the inertia and stiffness forces, the state variables relevant to the determination of damping forces are often not clear, (b) the spatial location of the damping sources are generally uncertain, although often structural joints are more responsible for the energy dissipation than the (solid) material, (c) the functional form of the damping model is difficult to establish experimentally, and (d) even if one manages to address the previous issues, the parameters that should be used

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in a chosen model are difficult to estimate. Thus the modelling of damping from first principles is very difficult, if not impossible, for complex engineering structures. For many years, engineers have bypassed these problems by using the viscous damping model, and a vast literature is available for this approach. In the viscous damping model, the instantaneous generalized velocities are assumed to be the only relevant state variables that determine the damping force, and for lightly damped structures this approach works reasonably well. Viscous damping is by no means the only damping model within the scope of linear analysis. Any causal model that makes the energy dissipation functional non-negative is a possible candidate for a damping model. Non-viscous damping models in general have more parameters and therefore are more likely to have a better match with experimental measurements. Adhikari (2002) and Wagner and Adhikari (2003) gave more background and references on non-viscous damping models in which the damping forces are assumed to depend on velocity time histories as well as the instantaneous velocities.

The damping model may also be non-local, where the damping force at any point on the beam depends on the velocity of the beam over a region (Flugge, 1975; Lei et al., 2006). Such models are a generalization of viscous damping, and examples include structures with viscoelastic damping layer treatments, structures supported on viscoelastic foundations, long adhesive joints in composite structures and surface damping treatments for vibration suppression using fluids (Ghoneim, 1997). Russell (1992) proposed a non-local damping model for the vibration analysis of a composite beam with an internal damping torque, and Ahmadi (1975) suggested a model of non-local viscoelasticity. Banks and Inman (1991) considered four different damping models for composite beams, namely viscous air damping, Kelvin–Voigt damping, time hysteresis damping and spatial hysteresis damping. The spatial hysteresis damping model may be treated as a non-local damping model and its non-local parameters estimated. The spatial hysteresis model, combined with viscous air damping, gave the best quantitative agreement with experimental time histories. Adhikari and Woodhouse (2001a,b) used a non-viscous damping model in the context of damping identification from measured transfer functions. Lin and Russell (2001) investigated the convergence of the bending moment of an elastic beam with spatial hysteresis damping.

Adhikari et al. (in press) presented a closed form solution for a beam with non-local damping using a transfer function method for the distributed parameter system (Yang and Tan, 1992). However, solutions were only possible for special cases of the spatial kernel function, and the influence of different kernel functions on the dynamic characteristics were not investigated. Lei et al. (2006) analysed systems with several kernel functions that describe possible models of the non-local effect of material damping using the Galerkin method. Recently the authors presented a finite element method for beams resting on non-local foundations (Friswell et al., in press). The purpose of this paper is to extend the method to beams with internal non-local damping.

2. The non-local beam model

Suppose that a beam of length L has non-local internal damping between x_1 and x_2, as shown in Fig. 1. The equation of motion for this beam may be expressed as the following integro-partial-differential equation,

$$\frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 w(x,t)}{\partial x^2}\right) + \rho A(x)\frac{\partial^2 w(x,t)}{\partial t^2} + Q_N(x,t) = f(x,t), \tag{1}$$

where $f(x,t)$ is excitation force, $EI(x)$ is the bending stiffness, $\rho A(x)$ is the mass per unit length of the beam and w is the transverse displacement of the beam. The beam is initially assumed to be at rest and standard boundary conditions are applied at the two ends. The internal damping model is an extension of that given by Sorrentino et al. (2003). Thus the damping force $Q_N(x,t)$ is given by

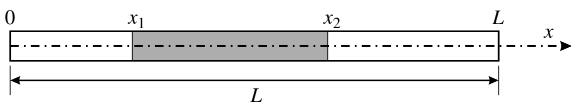

Fig. 1. A beam with partial non-local internal damping.

$$Q_N(x,t) = \frac{\partial^2}{\partial x^2}\left(\int_0^L \int_{-\infty}^t C(x,\xi,t-\tau)\frac{\partial^2 \dot{w}(\xi,\tau)}{\partial \xi^2}\,\mathrm{d}\tau\,\mathrm{d}\xi\right), \tag{2}$$

where the spatial integrations are over the region of non-local internal damping, whose extent is denoted by x_1 and x_2 in Fig. 1. $C(x,\xi,t)$ is the damping kernel and the dot denotes differentiation with respect to time. The boundary conditions for Eq. (1) are identical to an undamped beam for a clamped boundary. However, other boundary conditions, where the force and/or moment are zero, require the inclusion of the non-local effects in the force and/or moment. Since the thrust of this paper is finite element analysis, these issues are not considered further.

Eq. (1) is an integro-differential equation, and obtaining closed form solutions is difficult. Adhikari et al. (in press) presented a closed-form solution for beams with non-local damping in the foundation by the transfer function method. This approach is extended to internal local damping in Appendix A. Lei et al. (2006) presented approximate solutions using a Galerkin method for uniform beams with typical kernel functions. To treat more complicated problems with variable foundation stiffness, non-uniform section properties or with intermediate supports, a finite element method is developed.

2.1. The origin of non-local effects

Thus far the spatial kernel function has been specified, and no indication for its origin has been given. One would expect that the function decays monotonically, so that the effect of remote displacements on the force at a point reduces with distance. Exponential and Gaussian functions have this property and hence are often used as kernel functions (Eringen, 1987), and will be considered further in Section 2.2. Physically the origin of the non-local effects appears to be related to effects that the one dimensional beam equation is unable to capture. For foundation models extra degrees of freedom have been incorporated into models to allow for these effects, for example the model proposed by Kerr (1965). Two dimensional models of the foundation have been used to obtain parameters for the Kerr model (Avramidis and Morfidis, 2006). Friswell and Adhikari (2007) estimated non-local kernel functions directly from a two dimensional finite element model of a foundation, and kernel functions were fitted based on the sum of two exponential terms.

2.2. Typical kernel functions

In this paper we assume that the damping kernel function $C(x,\xi,t)$ is separable in space and time. Thus the kernel function takes the form

$$C(x,\xi,t-\tau) = C_0[H(x-x_1)-H(x-x_2)]\times[H(\xi-x_1)-H(\xi-x_2)]c(x-\xi)g(t-\tau). \tag{3}$$

The terms involving the Heaviside step function, $H(\cdot)$, are required to ensure that the kernel function is zero if either x or ξ are outside the region of non-local damping. This model represents non-local viscoelastic damping. Viscous damping is a special case where $g(t-\tau)=\delta(t-\tau)$, and local damping has $c(x-\xi)=\delta(x-\xi)$.

The function $g(t)$ is the relaxation kernel function of the non-viscous element and is often approximated as (Friswell et al., in press)

$$g(t) = \sum_{i=1}^{m} \frac{g_i}{\tau_i}\mathrm{e}^{-t/\tau_i}, \tag{4}$$

where g_i and τ_i are positive constants representing the damping coefficients and relaxation times, respectively. Alternatively fractional derivative or GHM models could be used (Bagley and Torvik, 1985; Kim and Kim, 2004; Golla and Hughes, 1985; McTavish and Hughes, 1993; Friswell et al., 1997).

Common choices for the spatial kernel function are the exponential decay,

$$c(x-\xi) = \frac{\alpha}{2}\mathrm{e}^{-\alpha|x-\xi|}, \tag{5}$$

and the Gaussian (error) function,

$$c(x-\xi) = \frac{\alpha}{\sqrt{2\pi}} e^{-\alpha^2(x-\xi)^2/2}, \tag{6}$$

although other models are also possible (Lei et al., 2006). If $\alpha \to \infty$ then one obtains the standard local model.

3. The finite element model

Generally the approach adopted when developing models using finite element analysis is to approximate the deformation within an element using nodal values of displacement and rotation. The kinetic and strain energy for each element is then computed and the contributions of each element are summed to obtain a global model of the structure. The damping matrix is obtained in a similar way using the dissipation function. One key aspect of finite element analysis is that the contributions from each element only depend on the displacements and rotations at the nodes associated with that element. Clearly for non-local damping this will not be the case, although the form of the exponential kernel function given by Eq. (5) does lead to a considerable simplification. Only Euler–Bernoulli beam theory is considered in this paper, although the formulation given is easily extended to Timoshenko beam theory.

A standard beam element is modelled using two nodes (at the ends of the beam element), and two degrees of freedom per node (translation and rotation). The deformation within the eth element, $w_e(\eta, t)$, is approximated using the standard cubic shape functions (Friswell et al., in press), $\mathbf{n}_e(\eta)$, for $\eta \in [0, \ell_e]$, as

$$w_e(\eta, t) = \mathbf{n}_e^T(\eta)\mathbf{q}_e(t), \tag{7}$$

where the vector $\mathbf{q}_e(t) = [w_{e1}(t)\, \psi_{e1}(t)\, w_{e2}(t)\, \psi_{e2}(t)]^T$ contains the nodal displacements (see Fig. 2).

The mass and stiffness matrices for Euler–Bernoulli beams are obtained in the usual way from the kinetic and strain energy. For non-local damping the global damping matrix is obtained using the dissipation function. The dissipation function will be obtained for a local viscous damping model initially. This will then be extended to a non-local viscous damping model and the corresponding damping matrix derived. Finally the viscoelastic material model will be included in the resulting equations of motion. The dissipation function for uniform local viscous damping, using the model of Sorrentino et al. (2003), is

$$\begin{aligned} \mathcal{F}(t) &= \frac{1}{2}\int_0^L \dot{w}(x,t)\frac{\partial^2}{\partial x^2}\left(C_0 \frac{\partial^2 \dot{w}(x,t)}{\partial x^2}\right)\mathrm{d}x \\ &= \frac{1}{2}\int_0^L C_0 \frac{\partial^2 \dot{w}(x,t)}{\partial x^2}\frac{\partial^2 \dot{w}(x,t)}{\partial x^2}\mathrm{d}x. \end{aligned} \tag{8}$$

Eq. (8) requires integration by parts, and the extra terms generated are zero because of the beam boundary conditions. The kernel function for local damping is a Dirac delta function, $\delta(x-\xi)$, and thus Eq. (8) may be written as

$$\mathcal{F}(t) = \frac{1}{2}\int_0^L \int_0^L C_0 \delta(x-\xi)\frac{\partial^2 \dot{w}(\xi,t)}{\partial \xi^2}\frac{\partial^2 \dot{w}(x,t)}{\partial x^2}\mathrm{d}\xi\,\mathrm{d}x. \tag{9}$$

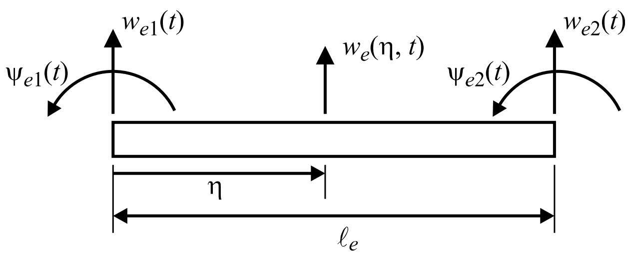

Fig. 2. The degrees of freedom of a beam finite element.

This motivates the extension to the non-local viscous damping case, where the dissipation function becomes

$$\mathcal{F}(t) = \frac{1}{2}\int_0^L \int_0^L C_0 c(x-\xi)\frac{\partial^2 \dot{w}(\xi,t)}{\partial \xi^2}\frac{\partial^2 \dot{w}(x,t)}{\partial x^2}\mathrm{d}\xi\,\mathrm{d}x. \tag{10}$$

If the region of the beam with non-local damping is split into M elements then the dissipation function of Eq. (10) may be written as

$$\mathcal{F}(t) = \frac{1}{2}\sum_{i,j=1}^{M} \dot{\mathbf{q}}_i(t)^{\mathrm{T}}\mathbf{C}_{ij}\dot{\mathbf{q}}_j(t), \tag{11}$$

where the element damping matrices are given by

$$\mathbf{C}_{ij} = C_0\int_0^{\ell_j}\int_0^{\ell_i} c(x_j+\hat{x}-x_i-\hat{\xi})\mathbf{n}_i''(\hat{\xi})\mathbf{n}_j''^{\mathrm{T}}(\hat{x})\mathrm{d}\hat{\xi}\,\mathrm{d}\hat{x}. \tag{12}$$

The primes denote spatial differentiation and $\hat{x}$ and $\hat{\xi}$ are local co-ordinates within the ith and jth element, for example $x = x_i + \hat{x}$, where x_i is the location of the ith node. The local nodal coordinates for the ith element are denoted $\mathbf{q}_i$.

In general each element damping matrix in Eq. (12) must be determined independently. The problem simplifies significantly if the beam is modelled using elements of equal length, so that $\mathbf{n}_i = \mathbf{n}$, $\ell_i = \ell$, $\forall i$, and the internal damping is uniform. Because the kernel function only involves $x_j - x_i$ the matrices $\mathbf{C}_{ij}$ are equal for a fixed $j - i$. Thus if the non-local damping covers M elements, only M of the element matrices $\mathbf{C}_{ij}$ have to be calculated. For general kernel functions the matrices must be calculated by numerical integration. However, the integrations may be performed explicitly for uniform damped beams with exponential or Gaussian (error function) kernels, as shown in Section 3.2 and 3.3.

3.1. Equations of motion and their solution

In general the degrees of freedom at all nodes will be coupled within a region where the structure has non-local damping behaviour. This means the global damping matrix will be full. The assembly process is straightforward and analogous to the approach used for the mass and stiffness matrices (Friswell et al., in press). The equation of motion for the free response of a beam with non-local viscous damping is of the form

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0}, \tag{13}$$

where $\mathbf{M}$, $\mathbf{K}$ and $\mathbf{C}$ are the global mass, stiffness and damping matrices with respect to the generalized co-ordinates of the beam model. Taking the Laplace transform gives

$$[s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}]\mathbf{q}(s) = \mathbf{0}. \tag{14}$$

A non-viscous material may be analysed by incorporating a relaxation kernel function into Eq. (14) as

$$[s^2\mathbf{M} + sG(s)\mathbf{C} + \mathbf{K}]\mathbf{q}(s) = \mathbf{0}. \tag{15}$$

Eq. (15) assumes that the time response of the viscoelastic foundation arises from a single material, leading to a single $G(s)$ term. If different materials are present then a damping term will appear for each different relaxation kernel, $G(s)$.

The eigenvalues are obtained by solving

$$\det|s^2\mathbf{M} + sG(s)\mathbf{C} + \mathbf{K}| = 0. \tag{16}$$

The corresponding mode shapes are obtained by substituting the eigenvalues into Eq. (15) and computing the null space of the matrix. Friswell et al. (in press) gave alternative solution strategies when $G(s)$ is a rational polynomial.

The eigenvalues obtained from the model will occur in complex conjugate pairs. In general the beam will not be classically or proportionally damped, and hence the eigenvectors will also be complex. The natural frequencies, ω_i, and damping ratios, ζ_i, may be obtained from the corresponding eigenvalues, λ_i, as follows

$$\omega_i = |\lambda_i| \quad \text{and} \quad \zeta_i = -\mathrm{real}(\lambda_i)/\omega_i. \tag{17}$$

In the examples the damping ratio will be used as a measure of modal damping.

3.2. Exponential kernel

Further simplification is possible for the exponential kernel. For a uniform beam and elements of equal length only two element matrices have to be calculated. The direct damping matrix is

$$\mathbf{C}_{ii} = \frac{C_0\alpha}{2}\int_0^\ell\int_0^\ell e^{-\alpha|\hat{x}-\hat{\xi}|}\mathbf{n}''(\hat{\xi})\mathbf{n}''^{\mathrm{T}}(\hat{x})\mathrm{d}\hat{\xi}\,\mathrm{d}\hat{x}. \tag{18}$$

If $j > i$ then

$$\mathbf{C}_{ij} = e^{-\alpha(j-i)\ell}\bar{\mathbf{C}}, \tag{19}$$

where the cross damping matrix is

$$\bar{\mathbf{C}} = \frac{C_0\alpha}{2}\int_0^\ell\int_0^\ell e^{-\alpha(\hat{x}-\hat{\xi})}\mathbf{n}''(\hat{\xi})\mathbf{n}''^{\mathrm{T}}(\hat{x})\mathrm{d}\hat{\xi}\,\mathrm{d}\hat{x}. \tag{20}$$

Similarly if $j < i$,

$$\mathbf{C}_{ij} = e^{-\alpha(i-j)\ell}\bar{\mathbf{C}}^{\mathrm{T}}. \tag{21}$$

Thus only two element damping matrices need to be computed and then the complete global damping matrix is easily derived. These matrices may be obtained explicitly by integrating Eqs. (18) and (20), and the resulting matrices are given in Appendix B.

3.3. Gaussian kernel

For a uniform beam, the error function (or Gaussian) kernel function given in Eq. (6) gives the direct damping matrix as

$$\mathbf{C}_{ii} = \frac{C_0\alpha}{\sqrt{2\pi}}\int_0^\ell\int_0^\ell e^{-\alpha^2(\hat{x}-\hat{\xi})^2/2}\mathbf{n}''(\hat{\xi})\mathbf{n}''^{\mathrm{T}}(\hat{x})\mathrm{d}\hat{\xi}\,\mathrm{d}\hat{x}. \tag{22}$$

This integral may be determined in closed form and is given in Appendix C.

For a uniform beam and elements of equal length, the cross damping matrix, $\mathbf{C}_{ij}$ depends on the value of $|j-i|$. Thus if $j > i$ and $d = j - i$ then

$$\mathbf{C}_{ij} = \frac{C_0\alpha}{\sqrt{2\pi}}\int_0^\ell\int_0^\ell e^{-\alpha^2(\hat{x}-\hat{\xi}+d\ell)^2/2}\mathbf{n}''(\hat{\xi})\mathbf{n}''^{\mathrm{T}}(\hat{x})\mathrm{d}\hat{\xi}\,\mathrm{d}\hat{x}. \tag{23}$$

Note also that

$$\mathbf{C}_{ij} = \mathbf{C}_{ji}^{\mathrm{T}} \tag{24}$$

and hence the cross damping matrices for $j < i$ are obtained immediately. The resulting closed form expressions are quite complex and are not given here. However these matrices are easily obtained using symbolic software.

4. Examples

4.1. A pinned–pinned beam

A pinned–pinned beam will be used as an example to demonstrate the use of the finite element methods for a partial non-local viscoelastically damped beam. The results will also be compared to the exact results from the transfer matrix method of Adhikari et al. (in press) and revised in Appendix A for internal damping. The dimensions of the beam are (see Fig. 1) $L = 200$ mm, $x_1 = 50$ mm, and $x_2 = 150$ mm. The Young's modulus is $E = 70$ G N/m^2, the density is $\rho = 2700$ kg/m^3, and the cross-section is 5×5 mm. Only one term is used in the relaxation kernel function given by Eq. (4), and the constants are defined as $\tau = \tau_1$ and $g_1 = g_\infty$. Unless stated otherwise, $C_0 = 0.01$ N s m^2 and $g_\infty = 1$.

Table 1 shows the discretisation errors in the first four eigenvalues predicted by the finite element model, with $\alpha = 1$ m^{-1} and $\tau = 0$. The results follow the usual trend, where the lower eigenvalues converge most quickly. Note that the effect of the damping is less on the even (asymmetric) modes, because the damping is located in the middle part of the beam where the curvature is less in the even modes. Table 2 shows the eigenvalues for various values of α and τ, and clearly the properties of the non-local viscoelastic damping have a significant effect on the eigenvalues.

Table 3 shows the first four eigenvalues for the Gaussian kernel with various values of α and τ, and these results should be compared to Table 2 for the exponential kernel. The Gaussian kernel model reduces the damping in all four modes, although the effect is more significant in the even (asymmetric) modes.

Fig. 3 shows the effect of varying the length of the part of the beam where the non-local effects occur. An exponential spatial damping kernel with $\alpha = 1$ m^{-1} is assumed and only the viscous time kernel ($\tau = 0$) is considered. The length of the non-local effect is defined as $x_2 - x_1$, and the non-local region is in the centre of the beam, so that $(x_1 + x_2)/2 = L/2$. The number of elements in the undamped regions of the beam is 4, and the number in the damped region is 8, giving a total of 16 elements. Modes 1 and 2 show increasing damping ratio with the length of the damping region. However modes 3 and 4 show a more complicated behaviour, due to the more complicated mode shapes.

Returning to the example with $x_1 = 50$ mm, and $x_2 = 150$ mm, the effect of the time constant of the viscoelastic kernel, τ, is now demonstrated. The finite element model with 8 elements and an exponential kernel with $\alpha = 1$ m^{-1} is used. Fig. 4 shows the effect of τ on the damping ratio, and highlights that the damping ratio changes significantly only when the time constant of the viscoelastic is close to or higher than the time constant of the mode of interest. As $\tau \to 0$ the viscoelastic model approaches the viscous model, and hence the damping

Table 1
The eigenvalues for the pinned–pinned beam with viscous damping and an exponential spatial kernel

FE (4 elements)	FE (8 elements)	Exact
$-178.49 \pm 1813.3j$	$-178.33 \pm 1812.8j$	$-178.34 \pm 1812.8j$
$-35.100 \pm 7282.4j$	$-35.108 \pm 7255.6j$	$-35.146 \pm 7253.7j$
$-1773.1 \pm 16870j$	$-1678.2 \pm 16578j$	$-1669.7 \pm 16556j$
$-571.00 \pm 32201j$	$-428.78 \pm 29128j$	$-424.45 \pm 29013j$

Table 2
The eigenvalues for the pinned–pinned beam obtained from the finite element method with 8 elements for the exponential kernel

$\tau = 0$, $\alpha = 1$ m^{-1}	$\tau = 0.001$ s, $\alpha = 1$ m^{-1}	$\tau = 0.001$ s, $\alpha = 10$ m^{-1}	$\tau = 0$, $\alpha = 10$ m^{-1}
$-178.33 \pm 1812.8j$	$-37.926 \pm 1887.9j$	$-165.23 \pm 2320.9j$	$-1976.1 \pm 2963.2j$
$-35.108 \pm 7255.6j$	$-0.65189 \pm 7260.1j$	$-35.457 \pm 7556.1j$	$-1674.6 \pm 8825.9j$
$-1678.2 \pm 16578j$	$-6.0185 \pm 16442j$	$-55.309 \pm 17354j$	$-16321 \pm 14570j$
$-428.78 \pm 29128j$	$-0.50395 \pm 29143j$	$-34.340 \pm 30197j$	$-43969 \pm 27909j$

Table 3
The eigenvalues for the pinned–pinned beam obtained from the finite element method with 8 elements for the Gaussian kernel

$\tau = 0$, $\alpha = 1$ m^{-1}	$\tau = 0.001$ s, $\alpha = 1$ m^{-1}	$\tau = 0.001$ s, $\alpha = 10$ m^{-1}	$\tau = 0$, $\alpha = 10$ m^{-1}
$-146.52 \pm 1813.1j$	$-31.706 \pm 1874.8j$	$-167.40 \pm 2344.6j$	$-1783.6 \pm 3151.5j$
$-0.0068 \pm 7255.4j$	$-0.0001 \pm 7255.4j$	$-0.1234 \pm 7256.3j$	$-6.6838 \pm 7255.6j$
$-1322.4 \pm 16519j$	$-4.7750 \pm 16421j$	$-36.896 \pm 17052j$	$-1813.6 \pm 21886j$
$-0.0074 \pm 29128j$	$-0.0000 \pm 29128j$	$-0.0085 \pm 29129j$	$-7.1143 \pm 29130j$

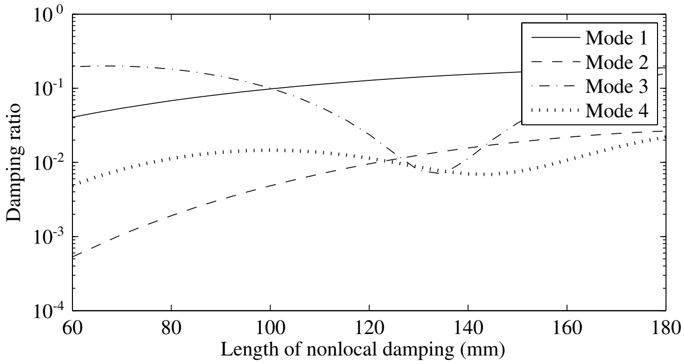

Fig. 3. The damping ratios of the pinned–pinned beam as the length of the damping region varies.

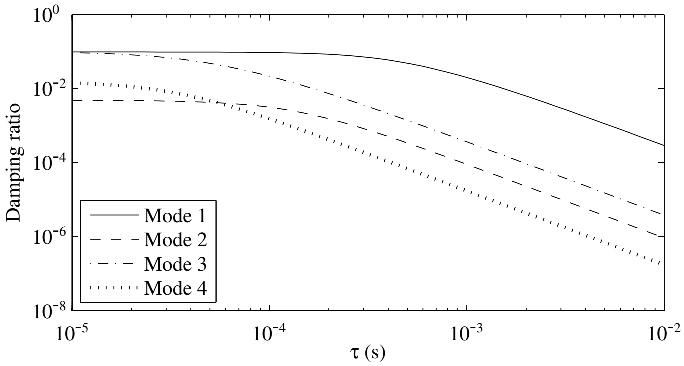

Fig. 4. The damping ratios of the pinned–pinned beam as the time constant of the viscoelastic kernel, τ, varies.

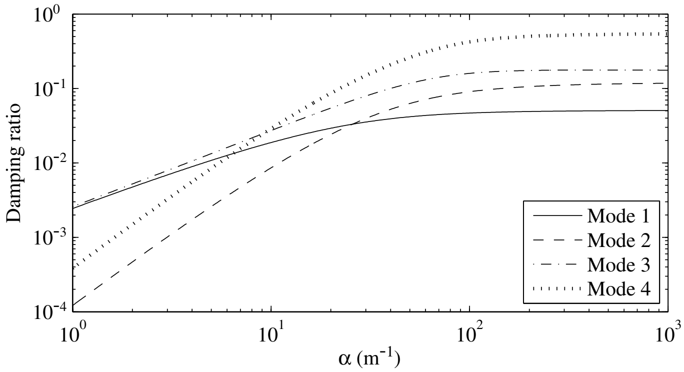

Fig. 5. The damping ratios of the pinned–pinned beam as the non-local constant α varies.

ratios converge to the viscous results. Fig. 5 shows the effect of varying the spatial constant in the exponential kernel of the non-local damping model, α, on the damping ratios of the first four modes. The damping amplitude constant, C_0, has been reduced to 0.00025 N s m^2 to ensure that the first four modes are all under-damped for all values of α. For large α the damping model approximates the local damping, and hence the damping ratios converge to those of the local model. The effect of α on modes 3 and 4 is much greater than

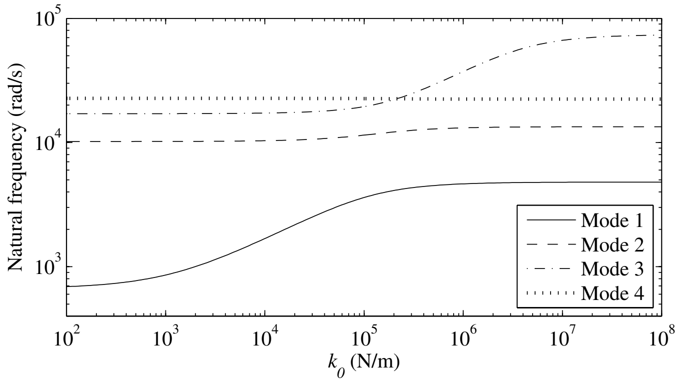

Fig. 6. The natural frequencies for the cantilever beam as the spring stiffness at the free end varies.

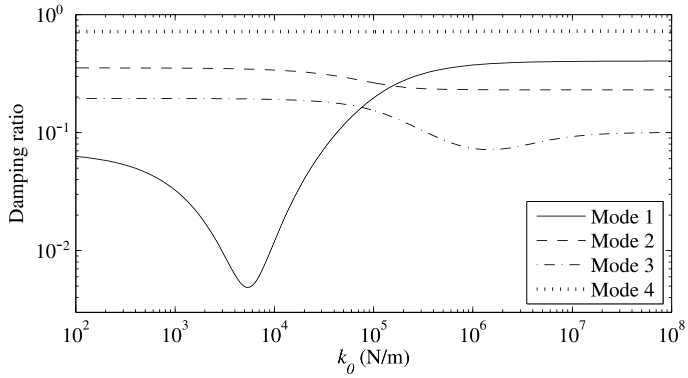

Fig. 7. The damping ratios for the cantilever beam as the spring stiffness at the free end varies.

on modes 1 and 2, and the value of α where the plots change slope is related to the reciprocal of the wavelength of the corresponding mode.

4.2. A cantilever beam

The beam of the previous section is now clamped at the end $x = 0$ and has a translational spring of stiffness k_0 at the end $x = L$. The material constants, length and cross-sectional area are identical to the pinned–pinned beam, and the non-local damping region is defined by $x_1 = 50$ mm and $x_2 = 150$ mm. The damping constants for the exponential kernel are $C_0 = 0.01$ N s m^2 and $\alpha = 10$ m^{-1} and the viscous model is assumed ($\tau = 0$). The finite element model with 8 elements is used. Fig. 6 shows the natural frequencies as the spring stiffness changes and highlights that odd modes in the free-end case change the most, and indeed modes 3 and 4 cross. The mode numbering is based on the free-end modes. Fig. 7 shows the corresponding damping ratios.

5. Conclusion

Existing numerical methods to analyse structures with non-local damping, such as the Galerkin approach, use displacement models defined over the whole structure. Usually the finite element method is preferred because of its ability to easily model a wide range of complex structures and boundary conditions. In this paper, a new method of analysing beams with internal non-local damping has been proposed, utilising the advantages of the finite element method. Non-local internal damping models mean that the damping force at a given point depends on the time history of the velocities within a spatial domain. The finite element models for a spatial exponential kernel requires only two element matrices to obtain the global damping matrix, one for the cross element terms and one for the direct terms. The cross matrix to model non-local effects is a novel concept and these matrices are zero for local damping models. Numerical solutions have been obtained for beams with a variety of boundary conditions, and the effect of different damping constants investigated. It was demonstrated that the form of the non-local damping model has a significant impact on the dynamic characteristics of structures.

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Appendix A. The exact solution for the exponential kernel

This appendix extends the transfer matrix approach of Adhikari et al. (in press) to internal damping for uniform beams with an exponential spatial kernel. Taking the Laplace Transform of Eqs. (1) and (2) gives,

$$EI\,W^{\mathrm{IV}}(x,s)+s^2\rho A\,W(x,s)+\frac{\alpha}{2}sG(s)\frac{\partial^2}{\partial x^2}\int_{x_1}^{x_2}\exp(-\alpha|x-\xi|)\frac{\partial^2 W(\xi,s)}{\partial\xi^2}\mathrm{d}\xi=0, \tag{A.1}$$

for $x\in(x_1,x_2)$. Here s is the complex Laplace parameter, $W(x,s)$ is the Laplace transform of $w(x,t)$. The roman superscripts, for example $(\bullet)^{\mathrm{IV}}$, denote the order of derivative with respect to the spatial variable x. Following the approach of Adhikari et al. (in press), this equation may be differentiated twice and the integral eliminated to give the sixth order ordinary differential equation

$$EI\,W^{\mathrm{VI}}(x,s)+s^2\rho A\,W^{\mathrm{II}}(x,s)-\alpha^2[EI\,W^{\mathrm{IV}}(x,s)+s^2\rho A\,W(x,s)]-\alpha^2 sG(s)\,W^{\mathrm{VI}}(x,s)=0. \tag{A.2}$$

The only difference between internal and foundation damping is the order of the derivative on the last term in Eq. (A.2). The solution procedure now follows that of Adhikari et al. (in press), but where the definition of $\mathbf{\Phi}$ is revised based on Eq. (A.2).

Appendix B. The damping matrices for the exponential kernel

This appendix gives the explicit expressions for the direct and cross stiffness matrices for the exponential damping kernel. The direct damping matrix is given by Eq. (18). The integral over $\hat{\xi}$ involving the term $|\hat{x}-\hat{\xi}|$ can be expressed as

$$\mathbf{C}_{ii}=\frac{C_0\alpha}{2}\int_{\hat{x}=0}^{\ell}\left\{\int_{\hat{\xi}=0}^{x}\mathrm{e}^{-\alpha(\hat{x}-\hat{\xi})}\mathbf{n}''(\hat{\xi})\mathbf{n}''^{\mathrm{T}}(\hat{x})\mathrm{d}\hat{\xi}+\int_{\hat{\xi}=x}^{\ell}\mathrm{e}^{\alpha(\hat{x}-\hat{\xi})}\mathbf{n}''(\hat{\xi})\mathbf{n}''^{\mathrm{T}}(\hat{x})\mathrm{d}\hat{\xi}\right\}\mathrm{d}\hat{x}. \tag{B.1}$$

Evaluating the above integral and simplifying we have

$$\mathbf{C}_{ii}=\frac{2C_0}{\ell^6\alpha^3}\mathbf{A}, \tag{B.2}$$

where the elements of the symmetric matrix $\mathbf{A}$ can be obtained in closed-form as

$$
\begin{aligned}
A_{11} &= 72 - 18\alpha^2\ell^2 + 6\alpha^3\ell^3 - (72 + 72\alpha\ell + 18\alpha^2\ell^2)e^{-\alpha\ell} \\
A_{12} &= 3\ell(12 - 3\alpha^2\ell^2 + \alpha^3\ell^3) - 9\ell(4 + 4\alpha\ell + \alpha^2\ell^2)e^{-\alpha\ell} \\
A_{13} &= -A_{11} \\
A_{14} &= A_{12} \\
A_{22} &= \ell^2(18 - 5\alpha^2\ell^2 + 2\alpha^3\ell^3) - 2\ell^2(9 + 9\alpha\ell + 2\alpha^2\ell^2)e^{-\alpha\ell} \\
A_{23} &= -A_{12} \\
A_{24} &= \ell^2(18 - 4\alpha^2\ell^2 + \alpha^3\ell^3) - \ell^2(18 + 18\alpha\ell + 5\alpha^2\ell^2)e^{-\alpha\ell} \\
A_{33} &= A_{11} \\
A_{34} &= -A_{12} \\
A_{44} &= A_{22}.
\end{aligned}
$$

The cross damping matrix in Eq. (20) is given by

$$
\bar{\mathbf{C}} = \frac{C_0\alpha}{2}\left(\int_0^\ell e^{\alpha\hat{\xi}}\mathbf{n}''(\hat{\xi})d\hat{\xi}\right)\left(\int_0^\ell e^{-\alpha\hat{x}}\mathbf{n}''^{\mathrm{T}}(\hat{x})d\hat{x}\right). \tag{B.3}
$$

Evaluating the above integral and simplifying we have

$$
\bar{\mathbf{C}} = \frac{2C_0}{\ell^6\alpha^3}\mathbf{v}_p\mathbf{v}_n^{\mathrm{T}}, \tag{B.4}
$$

where the elements of the vectors $\mathbf{v}_p$ and $\mathbf{v}_n$ are

$$
\begin{aligned}
\mathbf{v}_{p1} &= 6 + 3\alpha\ell - (6 - 3\alpha\ell)e^{\alpha\ell} \\
\mathbf{v}_{p2} &= \ell\left[3 + 2\alpha\ell - (3 - \alpha\ell)e^{\alpha\ell}\right] \\
\mathbf{v}_{p3} &= -6 - 3\alpha\ell + (6 - 3\alpha\ell)e^{\alpha\ell} \\
\mathbf{v}_{p4} &= \ell\left[3 + \alpha\ell - (3 - 2\alpha\ell)e^{\alpha\ell}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{v}_{n1} &= 6 - 3\alpha\ell - (6 + 3\alpha\ell)e^{-\alpha\ell} \\
\mathbf{v}_{n2} &= \ell\left[3 - 2\alpha\ell - (3 + \alpha\ell)e^{-\alpha\ell}\right] \\
\mathbf{v}_{n3} &= -6 + 3\alpha\ell + (6 + 3\alpha\ell)e^{-\alpha\ell} \\
\mathbf{v}_{n4} &= \ell\left[3 - \alpha\ell - (3 + 2\alpha\ell)e^{-\alpha\ell}\right].
\end{aligned}
$$

Appendix C. The damping matrices for the Gaussian kernel

For the Gaussian kernel the direct damping matrix can be obtained using

$$
\mathbf{C}_{ii} = \frac{C_0\alpha}{\sqrt{2\pi}}\int_0^\ell\int_0^\ell e^{-\alpha^2(x-\xi)^2/2}\mathbf{n}''(\hat{\xi})\mathbf{n}''^{\mathrm{T}}(\hat{x})d\hat{\xi}\,d\hat{x}. \tag{C.1}
$$

Evaluating the integral, we can express $\mathbf{C}_{ii}$ as

$$
\mathbf{C}_{ii} = \frac{\mathbf{A}\,\mathrm{erf}\left(\sqrt{2}\alpha\ell/2\right)}{\ell^3} + \frac{\sqrt{2}\mathbf{B}e^{-\alpha^2\ell^2/2}}{\alpha^3\ell^6\sqrt{\pi}} + 2\frac{\sqrt{2}\mathbf{D}}{\alpha^3\ell^6\sqrt{\pi}}, \tag{C.2}
$$

where the symmetric matrices **A**, **B** and **D** are given by

$$\mathbf{A} = \begin{bmatrix} 12 & 6\ell & -12 & 6\ell \\ 6\ell & 4\ell^2 & -6\ell & 2\ell^2 \\ -12 & -6\ell & 12 & -6\ell \\ 6\ell & 2\ell^2 & -6\ell & 4\ell^2 \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} -48 + 12\ell^2\alpha^2 & 6\ell(-4 + \ell^2\alpha^2) & 48 - 12\ell^2\alpha^2 & 6\ell(-4 + \ell^2\alpha^2) \\ 6\ell(-4 + \ell^2\alpha^2) & 4\ell^2(-3 + \ell^2\alpha^2) & -6\ell(-4 + \ell^2\alpha^2) & 2\ell^2(-6 + \ell^2\alpha^2) \\ 48 - 12\ell^2\alpha^2 & -6\ell(-4 + \ell^2\alpha^2) & -48 + 12\ell^2\alpha^2 & -6\ell(-4 + \ell^2\alpha^2) \\ 6\ell(-4 + \ell^2\alpha^2) & 2\ell^2(-6 + \ell^2\alpha^2) & -6\ell(-4 + \ell^2\alpha^2) & 4\ell^2(-3 + \ell^2\alpha^2) \end{bmatrix}$$

and

$$\mathbf{D} = \begin{bmatrix} 24 - 18\ell^2\alpha^2 & -3\ell(-4 + 3\ell^2\alpha^2) & -24 + 18\ell^2\alpha^2 & -3\ell(-4 + 3\ell^2\alpha^2) \\ -3\ell(-4 + 3\ell^2\alpha^2) & -\ell^2(-6 + 5\ell^2\alpha^2) & 3\ell(-4 + 3\ell^2\alpha^2) & 2\ell^2(3 - 2\ell^2\alpha^2) \\ -24 + 18\ell^2\alpha^2 & 3\ell(-4 + 3\ell^2\alpha^2) & 24 - 18\ell^2\alpha^2 & 3\ell(-4 + 3\ell^2\alpha^2) \\ -3\ell(-4 + 3\ell^2\alpha^2) & 2\ell^2(3 - 2\ell^2\alpha^2) & 3\ell(-4 + 3\ell^2\alpha^2) & -\ell^2(-6 + 5\ell^2\alpha^2) \end{bmatrix}.$$

The cross damping matrix for the Gaussian kernel is quite complex and the expressions are not given here.

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