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New Modification of Fixed Point Iterative Method for Solving Nonlinear Equations

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Abstract

In this paper, we have modified fixed point method and have established two new iterative methods of order two and three. We have discussed their convergence analysis and comparison with some other existing iterative methods for solving nonlinear equations.

Keywords

Modifications, Fixed Point Method, Nonlinear Equations

1. Introduction

In recent much attention has been given to establish new higher order iteration schemes for solving nonlinear equations. Many iteration schemes have been established by using Taylor series, Adomain decomposition, Homotopy pertrubation technique and other decomposition techniques [1]-[6]. We shall modify the fixed point method using taylor series on the functional equation $x = g(x)$ of nonlinear equation $f(x) = 0$. Initially, we do not put any restrictions on the original function *f*. In fixed point method, we rewrite $f(x) = 0$ as $x = g(x)$ where

1) There exist $[a,b]$ such that $g(x) \in [a,b]$ for all $x \in [a,b]$,
2) There exist $[a,b]$ such that $|g(x)| \le \lambda < 1$ for all $x \in [a,b]$,

The order of convergence of a sequence of approximation is defined as:

Definition 1.1 [7] *Let the sequence* $\{x_n\}$ *converges to* α. *If there is a positive integer p and real number C such that*

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$$\lim_{n\to\infty}\left|\frac{x_{n+1}-\alpha}{(x_n-\alpha)^p}\right|=C$$

then p is order of convergence.

Theorem 1.2 (*see* [6]). *Suppose that* $\varphi\in C^p[a,b]$. *If* $\varphi^{(k)}(x)=0$, *for* $k=0,1,2,\cdots,m-1$ *and* $\varphi^{(m)}(x)\neq 0$, *then the sequence* $\{x_n\}$ *is of order m.*

2. New Iteration Scheme

Consider the nonlinear equation

$$f(x)=0 \tag{2.1}$$

we can rewrite the above equation as

$$x=g(x) \tag{2.2}$$

We suppose that α is a root of (2.1) and γ is initial guess close to α. We can rewrite Equation (2.2) by using Taylor's expansion as:

$$x=g(\gamma)+(x-\gamma)g'(\gamma)+\frac{(x-\gamma)^2}{2!}g''(\gamma)+\cdots \tag{2.3}$$

if we truncate Equation (2.3) after second term then, we obtained

$$x=\frac{g(\gamma)-\gamma g'(\gamma)}{1-g'(\gamma)}$$

From above formulation we suggest the following algorithm for solving nonlinear Equation (2.1).

In algorithem form, we can write

$$x_{n+1}=\frac{g(x_n)-x_n g'(x_n)}{1-g'(x_n)}$$

we approximate

$$g'(x_n)\approx\frac{g(x_{n+1})-g(x_n)}{x_{n+1}-x_n}$$

Thus

$$x_{n+1}=\frac{g(x_n)-x_n\dfrac{g(x_{n+1})-g(x_n)}{x_{n+1}-x_n}}{1-\dfrac{g(x_{n+1})-g(x_n)}{x_{n+1}-x_n}}=\frac{-x_n g(x_{n+1})+x_{n+1}g(x_n)}{2g(x_n)-g(x_{n+1})-x_n}$$

if we take

$$x_{n+1}=g(x_n)$$

then we have the following algorithem;

Algorithm 2.1 *For a given* x_0, *we approximation solution* x_{n+1} *by the iteration scheme*:

$$x_{n+1}=\frac{-x_n g(g(x_n))+(g(x_n))^2}{2g(x_n)-g(g(x_n))-x_n}$$

If we truncate Equation (2.3) after third term then we have

$$x=g(\gamma)+(x-\gamma)g'(\gamma)+\frac{(x-\gamma)^2}{2!}g''(\gamma)$$

$$x=\frac{g(\gamma)-\gamma g'(\gamma)}{1-g'(\gamma)}+\frac{(x-\gamma)^2}{2!(1-g'(\gamma))}g''(\gamma)$$

$$x=\frac{g(\gamma)-\gamma g'(\gamma)}{1-g'(\gamma)}+\frac{g''(\gamma)}{2(1-g'(\gamma))}\left(\frac{g(\gamma)-\gamma g'(\gamma)}{1-g'(\gamma)}-\gamma\right)^2$$

$$x=\frac{g(\gamma)-\gamma g'(\gamma)}{1-g'(\gamma)}+\frac{g''(\gamma)(g(\gamma)-\gamma)^2}{2(1-g'(\gamma))^3}$$

In algorithem form, we can write

$$x_{n+1}=\frac{g(x_n)-x_n g'(x_n)}{1-g'(x_n)}+\frac{g''(x_n)(g(x_n)-x_n)^2}{2(1-g'(x_n))^3}$$

we approximate

$$g''(x_n)\approx\frac{g'(x_{n+1})-g'(x_n)}{x_{n+1}-x_n}$$

By substituting in above, we have

$$x_{n+1}=\frac{g(x_n)-x_n g'(x_n)}{1-g'(x_n)}+\frac{\frac{g'(x_{n+1})-g'(x_n)}{x_{n+1}-x_n}(g(x_n)-x_n)^2}{2(1-g'(x_n))^3}$$

$$=\frac{g(x_n)-x_n g'(x_n)}{1-g'(x_n)}+\frac{(g'(g(x_n))-g'(x_n))(g(x_n)-x_n)}{2(1-g'(x_n))^3}$$

Thus, we have the following algorithem;

Algorithm 2.2 *For a given* x_0, *we approximation solution* x_{n+1} *by the iteration scheme*:

$$x_{n+1}=\frac{g(x_n)-x_n g'(x_n)}{1-g'(x_n)}+\frac{(g'(g(x_n))-g'(x_n))(g(x_n)-x_n)}{2(1-g'(x_n))^3}$$

3. Convegence Analysis

In this section, we discuss the convergence of Algorithm (2.1) and (2.2).

Theorem 3.1 *Let* $f: I\subset\mathbb{R}\to\mathbb{R}$ *for an open interval I and consider that the nonlinear equation* $f(x)=0$ (*or* $x=g(x)$) *has simple root* $\alpha\in I$, *where* $g(x): I\subset\mathbb{R}\to\mathbb{R}$ *be sufficiently smooth in the neighbourhood of the root* α. *If* x_0 *is sufficiently close to* α *then iteration scheme defined by Algorithm* 2.1 *has at least second order convergence.*

Proof. Let α be simple zero of $f(x)=0$ and $x=g(x)$ be its functional equation. Let e_n and e_{n+1} be errors at n^{th} and $(n+1)^{th}$ iterations respectively. Then expanding $g(x_n)$ and $g(g(x_n))$ about α, we have

$$g(x_n)=\alpha+e_n g'(\alpha)+\frac{1}{2}e_n^2 g''(\alpha)+\frac{1}{6}e_n^3 g'''(\alpha)+O(e_n^4) \tag{3.1}$$

and

$$g(g(x_n))=\alpha+e_n g''(\alpha)+\frac{1}{2}e_n^2\left(g'(\alpha)g''(\alpha)+g''(\alpha)(g'(\alpha))^2\right) + \frac{1}{6}e_n^3\left(g'(\alpha)g'''(\alpha)+g''(\alpha)(g'(\alpha))^2+g'''(\alpha)(g'(\alpha))^3\right)+O(e_n^4) \tag{3.2}$$

Algorithem (2.1) is given by

$$x_{n+1}=\frac{-x_n g\left(g\left(x_n\right)\right)+\left(g\left(x_n\right)\right)^2}{2g\left(x_n\right)-g\left(g\left(x_n\right)\right)-x_n}$$

By substituting values from Equations (3.1) and (3.2) in above, we get

$$x_{n+1}=\alpha+\frac{1}{2}\frac{g'(\alpha)g''(\alpha)}{g'(\alpha)-1}e_n^2+O\left(e_n^3\right)$$

$$e_{n+1}=\frac{1}{2}\frac{g'(\alpha)g''(\alpha)}{g'(\alpha)-1}e_n^2+O\left(e_n^3\right)$$

Hence algorithem (2.1) has second order convergence.

Theorem 3.2 *Let* $f:I\subset\mathbb{R}\to\mathbb{R}$ *for an open interval I and consider that the nonlinear equation* $f(x)=0$ (*or* $x=g(x)$) *has simple root* $\alpha\in I$, *where* $g(x):I\subset\mathbb{R}\to\mathbb{R}$ *be sufficiently smooth in the neighbourhood of the root* α. *If* x_0 *is sufficiently close to* α *then iteration scheme defined by Algorithm 2.2 has at least third order convergence.*

Proof. Let α be simple zero of $f(x)=0$ and $x=g(x)$ be its functional equation. Let e_n and e_{n+1} be errors at n^{th} and $(n+1)^{th}$ iterations respectively. Then expanding $g(x_n)$, $g'(x_n)$ and $g'\left(g\left(x_n\right)\right)$ about α, we have

$$g\left(x_n\right)=\alpha+e_n g'(\alpha)+\frac{1}{2}e_n^2 g''(\alpha)+\frac{1}{6}e_n^3 g'''(\alpha)+O\left(e_n^4\right) \tag{3.3}$$

$$g'\left(x_n\right)=\alpha g'(\alpha)+e_n g''(\alpha)+\frac{1}{2}e_n^2 g'''(\alpha)+\frac{1}{6}e_n^3 g^{iv}(\alpha)+O\left(e_n^4\right) \tag{3.4}$$

$$\begin{aligned} g'\left\{g\left(x_n\right)\right\}&=g'(\alpha)+e_n g'(\alpha)g''(\alpha)+\frac{1}{2}e_n^2\left[\left\{g''(\alpha)\right\}^2+\left\{g'(\alpha)\right\}^2 g'''(\alpha)\right] \\ &\quad+\frac{1}{6}e_n^3\left[g'''(\alpha)g''(\alpha)+g'''(\alpha)g^{iv}(\alpha)+3g'''(\alpha)g''(\alpha)g'(\alpha)\right]+O\left(e_n^4\right) \end{aligned} \tag{3.5}$$

Algorithem (2.2) is given by

$$x_{n+1}=\frac{g\left(x_n\right)-x_n g'\left(x_n\right)}{1-g'\left(x_n\right)}+\frac{\left(g'\left(g\left(x_n\right)\right)-g'\left(x_n\right)\right)\left(g\left(x_n\right)-x_n\right)}{2\left(1-g'\left(x_n\right)\right)^3}$$

By substituting values from Equations (3.3), (3.4) and (3.5) in above, we get

$$x_{n+1}=\alpha+\frac{-3\left\{g'(\alpha)\right\}^2 g'''(\alpha)+4g'(\alpha)g'''(\alpha)-g'''(\alpha)+6\left\{g''(\alpha)\right\}^2}{12\left(-1+g'(\alpha)\right)^2}e_n^3+O\left(e_n^4\right)$$

$$e_{n+1}=\frac{-3\left\{g'(\alpha)\right\}^2 g'''(\alpha)+4g'(\alpha)g'''(\alpha)-g'''(\alpha)+6\left\{g''(\alpha)\right\}^2}{12\left(-1+g'(\alpha)\right)^2}e_n^3+O\left(e_n^4\right)$$

Hence the order of convergence fo algorithm 2.2 is least 3.

4. Numerical Results

In this section, we present some example to make the comparitive study of fixed point method (FPM), Newton method (NM), Abbasbandy method (AM), Homeier method (HM), Chun method (CM), Householder method (HHM), Algorithem 2.1 and Algorithm 2.2 developed in this paper. We use $\epsilon=10^{-15}$. The following criterias are used for computer programs:

1) $\left|x_n-x_{n-1}\right|<\epsilon$

2) $\left|f\left(x_n\right)\right| < \epsilon$

We consider the following examples to illustarate the performance of our newly established iteration scheme.

$$f_1(x) = \sin^2 x - x^2 + 1$$

$$f_2(x) = x^2 - e^x - 3x + 2$$

$$f_3(x) = \cos x - x$$

$$f_4(x) = (x-1)^3 - 1$$

$$f_5(x) = x^3 - 10$$

$$f_6(x) = e^{x^2+7x-30} - 1$$

Comparison Table

Examples	Functional eq.	IT	x_n	$f(x_n)$
$f_1, x_0 = 1$	$g(x) = \sin x + \frac{1}{x + \sin x}$			
NM		7	1.404491648315341226350868177	$-1.04e^{-50}$
AM		5	1.404491648315341226350868177	$-5.81e^{-55}$
HM		4	1.404491648315341226350868178	$-5.4e^{-62}$
CM		5	1.404491648315341226350868178	$-2.0e^{-63}$
HHM		6	1.404491648215341226035086820	$1.81e^{-25}$
FPM		17	1.404491648215341226035086441	$9.33e^{-25}$
Alg. 2.1		4	1.404491648215341226035086817	$7.56e^{-33}$
Alg. 2.2		4	1.404491648215341226035086817	$9.31e^{-60}$
$f_2, x_0 = 2$	$g(x) = \frac{e^x - 2}{x - 3}$			
NM		6	0.257530285439860760455367303	$2.93e^{-55}$
AM		5	0.257530285439860760455367304	$1.0e^{-63}$
HM		5	0.257530285439860760455367305	0
CM		4	0.257530285439860760455367304	$1.0e^{-63}$
HHM		5	0.257530285439860760455367306	$-6.0e^{-24}$
FPM		56	0.257530285439860760455367015	$1.09e^{-24}$
Alg. 2.1		6	0.257530285439860763223411636	$3.31e^{-38}$
Alg. 2.2		5	0.2575302854398607604553673049	$4.85e^{-66}$
$f_3, x_0 = 1.7$	$g(x) = \cos x$			
NM		5	0.739085133215160641655372084	$-2.03e^{-32}$
AM		4	0.739085133215160641655372085	$-7.14e^{-47}$
HM		4	0.739085133215160641655372086	$-5.02e^{-59}$

Continued

CM		4	0.7390851332151606416553720 87	0
HHM		4	0.739085133215160641655372089	$3.92e^{-16}$
FPM		100	0.739085133215160645628855081	$6.65e^{-18}$
Alg. 2.1		5	0.739085133215160641655311945	$2.38e^{-25}$
Alg. 2.2		4	0.739085133215160641655312087	$1.98e^{-55}$
$f_4, x_0 = 3.5$	$g(x) = 1 + \sqrt{\frac{1}{x-1}}$			
NM		8	2.000000000000000000000000023	$2.06e^{-42}$
AM		5	2	0
HM		5	2	0
CM		5	2	0
HHM		7	2.0000000000000000000000008280390	$2.48e^{-21}$
FPM		81	1.999999999999999999999999621	$1.13e^{-24}$
Alg. 2.1		5	2.000000000000000000000000001	$4.25e^{-30}$
Alg. 2.2		4	2.000000000000000000000000000	$5.74e^{-53}$
$f_5, x_0 = 1.5$	$g(x) = \sqrt{\frac{10}{x}}$			
NM		7	2.154434690031883721759235664	$2.06e^{-54}$
AM		5	2.154434690031883721759235667	$-5.0e^{-63}$
HM		5	2.154434690031883721759235667	$-5.0e^{-63}$
CM		5	2.154434690031883721759235667	$-5.0e^{-63}$
HHM		6	2.154434690031883721759293567	$7.86e^{-27}$
FPM		80	2.154434690031883721759292921	$8.98e^{-24}$
Alg. 2.1		4	2.154434690031883721764919377	$7.83e^{-20}$
Alg. 2.2		4	2.154434690031883721759293567	$1.07e^{-53}$
$f_6, x_0 = 3.5$	$g(x) = \frac{1}{7}(30 - x^2)$			
NM		13	3.000000000000000000000000000	$1.52e^{-47}$
AM		7	3.000000000000000000000000000	$-4.33e^{-48}$
HM		8	3.000000000000000000000000000	$2.0e^{-62}$
CM		8	3.000000000000000000000000000	$2.0e^{-62}$
HHM		12	3.000000000000000000000000421	$5.48e^{-24}$
FPM		100	3.000000102855948739906642798	$1.33e^{-06}$
Alg. 2.1		4	2.999999999999999999999458108	$7.04e^{-22}$
Alg. 2.2		3	3.000000000000000000000000000	$1.93e^{-33}$

5. Conclusion

We have modified the fixed point method for solving nonlinear equations. We have established two new algorithms of convergence order two and three. We have solved some nonlinear equations to show the performance and efficiency of our newly developed iteration schemes. From comparison table, we conclude that these schemes perform much better than Newton method, Abbasbandy method, Chun method, Homeier method, Householder method etc.

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