POISSON PROCESSES WITH FUZZY RATE

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ABSTRACT

Poisson processes, particularly the time-dependent extension, play important roles in reliability and risk analysis. It should be fully aware that the Poisson modeling in the current reliability engineering and risk analysis literature is merely an ideology under which the random uncertainty governs the phenomena. In other words, current Poisson Models generate meaningful results if randomness assumptions hold. However, the real world phenomena are often facing the co-existence reality and thus the probabilistic Poisson modeling practices may be very doubtful. In this paper, we define the random fuzzy Poisson process, explore the related average chance distributions, and propose a scheme for the parameter estimation and a simulation scheme as well. It is expecting that a foundational work can be established for Poisson random fuzzy reliability and risk analysis.

1   INTRODUCTION

It should be fully aware that vagueness is an intrinsic feature in today's diversified business environments, just as Carvalho and Machado (2006) commented, "In a global market, companies must deal with a high rate of changes in business environment. … The parameters, variables and restrictions of the production system are inherently vagueness." Therefore the co-existence of random uncertainty and fuzzy uncertainty is inevitable reality of safety and reliability analysis and modelling.

It is a well-established fact that Poisson processes and particularly the non-stationary Poisson processes play important roles in safety and reliability modeling. Many researchers contributed to the probabilistic developments, see Crow (1974), Guo and Love (1992, 1994, 2004), Guo et al. (2007), Love and Guo (1991).

Logically, it is obvious that probabilistic modeling is only a good approximation to real world problem when random uncertainty governs the phenomenon. If fuzziness and randomness both appear then probabilistic modeling may be questionable. Therefore, developing the appropriate models for modeling fuzziness and randomness co-existence is necessary.

In this paper, we are trying to offer a systematic treatment for the random fuzzy Poisson processes not only in the mathematical sense (building models based on postulates and definitions) but also in the statistical sense (estimation and hypothesis testing based on sample data).

2   FOUNDATION OF RANDOM FUZZY VARIABLE

Without a solid understanding of the intrinsic feature of random fuzzy variable, there is no base for exploring the modelling of random fuzzy processes. Therefore, it is necessary to briefly review Liu's hybrid variable theory established on the axiomatic credibility measure and probability measure foundations.

First let us review the credibilistic fuzzy variable theory. Let Θ be a nonempty set, and $\mathsf{P}(\Theta)$ the power set on Θ. Each element, let us say, $A \subset \Theta$, $A \in \mathsf{P}(\Theta)$ is called an fuzzy event. A number denoted as $\mathrm{Cr}\{A\}$, $0 \leq \mathrm{Cr}\{A\} \leq 1$, is assigned to event $A \in \mathsf{P}(\Theta)$, which indicates the credibility grade with which event $A \in \mathsf{P}(\Theta)$ occurs. $\mathrm{Cr}\{A\}$ satisfies following axioms given by Liu (2004, 2007):

Axiom 1: $\mathrm{Cr}\{\Theta\} = 1$.

Axiom 2: $\mathrm{Cr}\{\cdot\}$ is non-decreasing, i.e., whenever $A \subset B$, $\mathrm{Cr}\{A\} \leq \mathrm{Cr}\{B\}$.

Axiom 3: $\mathrm{Cr}\{\cdot\}$ is self-dual, i.e., for any $A \in 2^{\Theta}$, $\mathrm{Cr}\{A\} + \mathrm{Cr}\{A^c\} = 1$.

Axiom 4: $\mathrm{Cr}\{\cup_i A_i\} = \sup_i \left[\mathrm{Cr}\{A_i\}\right]$ for any $\{A_i\}$ with $\mathrm{Cr}\{A_i\} \leq 0.5$.

Definition 1: (Liu (2004, 2007)) Any set function $\mathrm{Cr}: \mathsf{P}(\mathrm{Q}) \circledR [0,1]$ satisfies *Axioms 1-4* is called a credibility measure. The triple $(\mathrm{Q}, \mathsf{P}(\mathrm{Q}), \mathrm{Cr})$ is called the credibility measure space.

It should be fully aware that credibility measure only follows sub-σ-additive property, but probability measure does enjoy the σ-additive property. This character of credibility measure relaxes the assumptions of the set mapping so that it might cover a wider category of real world uncertain problems, but brings new difficulties in its mathematical treatments.

Definition 2: A fuzzy variable ξ is a measurable mapping, i.e., $\xi : (\Theta, \mathsf{P}(\Theta)) \to (\mathrm{R}, \mathsf{B}(\mathrm{R}))$. The measurable mapping is characterized by the membership of the pre-image of event $B = (-\Gamma, r]$ under fuzzy variable ξ to the power set $\mathsf{P}(\Theta)$. In other words,

$$\forall B \in \mathsf{B}(\mathrm{R}), \{\theta \in \Theta : \xi \in B\} \in \mathsf{P}(\Theta) \tag{1}$$

The measurability of fuzzy variable ξ definitely induces a measure on the measurable space $(\mathrm{R}, \mathsf{B}(\mathrm{R}))$. Let us denote the induced measure as $\mathfrak{m}^c$. For $\forall B \in \mathsf{B}(\mathrm{R})$, the induced measure is

$$\mu^c\{B\} = \mathrm{Cr}\{\theta \in \Theta : \xi \in B\} = \mathrm{Cr}\{\theta \in \Theta : \xi(\omega) \leq r\} \tag{2}$$

Therefore, further denote $\mu^c = \mathrm{Cr} \circ \xi^{-1}$ and specifically, the distribution is defined by the induced measure

$$\mathrm{L}(x) = \mathfrak{m}^c\{(-\Gamma, r]\} = \mathrm{Cr}\{q \in \mathrm{Q} : x(q) \jmath r\} \tag{3}$$

The induced distribution by fuzzy variable ξ is just the credibility distribution which characterizes the measurement of vague (or fuzzy) uncertainty associated with every event with fuzzy variable ξ.

Definition 3: (Liu (2004, 2007)) The credibility distribution $\Lambda : \mathrm{R} \to [0,1]$ of a fuzzy variable ξ on $(\mathrm{Q}, \mathsf{P}(\mathrm{Q}), \mathrm{Cr})$ is

$$\Lambda(x) = \mathrm{Cr}\{\theta \in \Theta \mid \xi(\theta) \leq x\} \tag{4}$$

Credibility measure, as an axiomatic measure development, the set class, power set $\mathsf{P}(\Theta)$ plays the critical roles in defining set function credibility measure Cr as well as the measurability of fuzzy variable. However, it is necessary to keep in mind that power set $\mathsf{P}(\Theta)$ is the largest s-algebra of space Q. The establishment of set function on power set inevitably brings different feature from that establishing probability measure on the smallest s-algebra $\mathsf{A}(\mathrm{W})$ of a space W.

Therefore, a fuzzy variable is not a fuzzy set in the sense of Zadeh's fuzzy theory (1965, 1978), in which a fuzzy set is defined by a membership function.

Liu (2004, 2007) defined a random fuzzy variable as a mapping from the credibility space $(Q, 2^Q, Cr)$ to a set of random variables. Again, we should be aware that a random fuzzy variable here takes real numbers as its values, which behaves very similar to a random variable. We would like to present an intuitive definition similar to that of stochastic process in probability theory and expect readers who are familiar with the basic concept of stochastic processes can understand the comparative definition.

Definition 4: ((Guo et al, 2007)) A random fuzzy variable, denoted as $\xi = \left\{ X_{\beta(\theta)}, \theta \in \Theta \right\}$, is a collection of random variables X_β defined on the common probability space $(\Omega, A, \Pr)$ and indexed by a fuzzy variable $\beta(\theta)$ defined on the credibility space $(Q, 2^Q, Cr)$.

Similar to the interpretation of a stochastic process, $X = \left\{ X_t, t \in \mathbf{R}^+ \right\}$, a random fuzzy variable is a bivariate mapping from $\left(\Omega \times \Theta, A \times 2^\Theta \right)$ to the space $\left(R, B(R) \right)$, where $B(R)$ denotes Borel σ-algebra on real number set $R \equiv (-\infty, \infty)$. As to the index, in stochastic process theory, index used is referred to as *time* typically, which is a positive (scalar variable), while in the random fuzzy variable theory, the "index" is a fuzzy number (i.e., variable), say, β. Using uncertain parameter as index is not starting in random fuzzy variable definition. In stochastic process theory we already know that the stochastic process $X = \left\{ X_{\tau(\omega)}, \omega \in \Omega \right\}$ uses stopping time $t(w)$, w OW, in which an (uncertain) random variable is used as its index.

In random fuzzy variable theory, there are different types of chances measures proposed for characterizing a random fuzzy variable. What we are going to use is the average chance measure, denoted as ch, which will plays a similar role to a probability measure, denoted as $\Pr$, in probability theory.

Definition 5: (Liu (2004, 2007)) Let x be a random fuzzy variable, then the average chance measure denoted by $ch\{\}$, of a random fuzzy event $\{x J\ x\}$, is

$$ch\{x J\ x\} = \int_0^1 Cr\left\{ q O Q | \Pr\{x(q) J\ x\} i\ a \right\} da \tag{5}$$

Then function $Y()$ is called as average chance distribution if and only if

$$Y(x) = ch\{x J\ x\} \tag{6}$$

It is quite important to emphasize here that the definition of random fuzzy variable is constructive. The mapping order is essential. The following theorem is actually a summary of Liu's definition and examples in his book. For example, if a random variable η has zero mean and a fuzzy variable ζ, then the sum of the two, $\eta + \zeta$, results in a random fuzzy variable ξ, Liu (2004, 2007). Accordingly, let $\eta \sqcup N(0, \sigma^2)$, i.e., a normal random variable with zero mean and variance σ^2, and let ζ be a triangular fuzzy number (i.e., variable), then $\xi = \eta + \zeta$ is a normal random fuzzy variable, denoted as $x \overset{d}{:} N(z, s^2)$. Liu (2004, 2007) also mentioned an example in which an exponential density, be^{-bt} having a fuzzy parameter b. We state Liu's ideas formally as a theorem.

Theorem 1: Let ζ be a fuzzy variable defined on the credibility space $(\Theta, P(\Theta), Cr)$ and τ be a random variable defined on the probability space $(\Omega, A(\Omega), P)$, then

(1) Let $\oplus$ be an arithmetic operator, which can be "$+$", "$-$", "$\times$" or "$\div$" operation, such that $\zeta \oplus \tau$ maps from $(\Theta, P(\Theta), Cr)$ to a collection of random variables on $(\Omega, A(\Omega), P)$, denoted by

ξ. Then ξ is a random fuzzy variable defined on hybrid product space $(\Theta, \mathsf{P}(\Theta), \mathrm{Cr}) \times (\Omega, \mathsf{A}(\Omega), P)$.

(2) Let $f : \mathrm{R} \times \mathrm{R} \to \mathrm{R}$ be a continuous, such that $f(\zeta, \tau)$ maps from $(\Theta, \mathsf{P}(\Theta), \mathrm{Cr})$ to a collection of random variables on $(\Omega, \mathsf{A}(\Omega), P)$, denoted by ξ. Then $\xi = f(\zeta, \tau)$ is a random fuzzy variable defined on hybrid product space $(\Theta, \mathsf{P}(\Theta), \mathrm{Cr}) \times (\Omega, \mathsf{A}(\Omega), P)$.

(3) Let $F(x;\theta)$ be the probability distribution of random variable τ with parameter θ (possible vector-valued), then $F(x;\zeta)$ defines a random fuzzy variable ξ on the hybrid product space $(\Theta, \mathsf{P}(\Theta), \mathrm{Cr}) \times (\Omega, \mathsf{A}(\Omega), P)$.

Note that the Theorem 1 is merely specifying three subfamilies of random fuzzy variables. Particularly, the Item (1) and (2) are strictly stating Liu's definition of random fuzzy variable, Liu (2004, 2007), for avoiding the possible confusion with general hybrid variable, particularly, fuzzy random variable. A straightforward example for Item (2) are linear function $f(\zeta, \tau) = \alpha_0 + \alpha_1 \zeta + \alpha_2 \tau$, $\alpha_0, \alpha_1, \alpha_2 \in \mathrm{R}$. Another example for Item (2) is $f(\zeta, \tau) = (\tau/\varsigma)^{\alpha}$, $\alpha > 0$. As the Item (3), it is a direct extension to Liu's exponential distribution with fuzzy rate parameter, Liu (2004, 2007).

3 RANDOM FUZZY POISSON PROCESSES

It is well-known fact that a Poisson random variable N takes nonnegative integer-value with probability:

$$\Pr\{N = k\} = \frac{1^k}{k!} e^{-1}, k = 0, 1, 2, \mathrm{L} \tag{7}$$

where $1 > 0$ is the parameter representing the rate of event occurrences.

3.1 Probabilistic Poisson processes

In stochastic process theory, Poisson process may be defined in different ways although revealed the same intrinsic features. Grimmett and Stirzaker (1992) stated a formal definition as following:

Definition 6: A Poisson process with intensity 1 is a process $N = \{N(t), t \ \mathrm{i}\ 0\}$ taking values in $\mathrm{S} = \{0, 1, 2, \mathrm{L}\}$ such that:

(a) $N(0) = 0$; if $s < t$, then $N(s) \mathrm{J}\ N(t)$;

(b) $\Pr\{N(t + h) = n + m \mid N(t) = n\}$

$$= \begin{cases} 1h + o(h) & \text{if } m = 1 \\ o(h) & \text{if } m > 1 \\ 1 - 1h + o(h) & \text{if } m = 0 \end{cases}$$

(c) if $s < t$ then number $N(t) - N(s)$ of an emission in the interval $(s, t]$ is independent of the times of emissions during $[0, s)$.

It is fairly straightforward that at any time t, $N(t)$ is a Poisson random variable with rate $1t$ with probability:

$$\Pr\{N(t) = k\} = \frac{(1t)^k}{k!} e^{-1t}, k = 0, 1, 2, \mathrm{L} \tag{8}$$

Associated with probabilistic Poisson processes, the critical fact is the distributions of the inter-arrival times which have many applications to reliability and risk analysis.

Theorem 2: The successive inter-arrival (sojourn) times in a Poisson process $N = \{N(t), t \geq 0\}$ with intensity λ are *i.i.d.* variables with common probability density function $\lambda e^{-\lambda t}$.

Theorem 3: The waiting time to the nth event, W_n, in a Poisson process $N = \{N(t), t \geq 0\}$ with intensity λ has a gamma distribution with probability density function $f_{W_n}(t) = \dfrac{(\lambda t)^{n-1}}{(n-1)!} \lambda e^{-\lambda t}$.

The proof of theorem 2 is an application of Poisson process definition, while the proof of theorem 3 is completed by noticing the fact that the distribution of waiting times is merely that of n *i.i.d.* exponential variables.

3.2   Random fuzzy stationary Poisson processes

According to Theorem 1, an intuitive formation of a random fuzzy Poisson process is to assume the intensity λ to be a credibilistic fuzzy variable defined on credibility space $(\Theta, \mathcal{P}(\Theta), Cr)$ with credibility distribution function Λ.

Definition 7: A random fuzzy Poisson process with credibilistic fuzzy intensity λ on credibility space $(\Theta, \mathcal{P}(\Theta), Cr)$ is a process $N = \{N(t), t \geq 0\}$ taking values in $S = \{0, 1, 2, \ldots\}$ such that:

(a) $N(0) = 0$; if $s < t$, then $N(s) \leq N(t)$;

(b) $\Pr\{N(t+h) = n+m \mid N(t) = n\}$

$$= \begin{cases} \lambda h + o(h) & \text{if } m = 1 \\ o(h) & \text{if } m > 1 \\ 1 - \lambda h + o(h) & \text{if } m = 0 \end{cases}$$

(c)   if $s < t$ then number $N(t) - N(s)$ of an emission in the interval $(s, t]$ is independent of the times of emissions during $[0, s)$.

It is obvious that in Definition 7 the intensity parameter λ is a credibilistic fuzzy parameter (credibilistic fuzzy variable, indeed). For a given value of parameter $\lambda = \lambda_0$, $N = \{N(t), t \geq 0\}$ is just a probabilistic Poisson process. However, if λ is a fuzzy parameter, then for any given time t, the count $N(t)$ is a random fuzzy variable according to Theorem 1. Therefore, Definition 7 defines a stationary random fuzzy Poisson process.

Theorem 4: The successive inter-arrival (sojourn) times in a random fuzzy Poisson process $N = \{N(t), t \geq 0\}$ with credibilistic fuzzy intensity λ having a piecewise linear credibility distribution

$$
L(x) = \begin{cases}
0 & x \leq a \\
\dfrac{x-a}{2(b-a)} & a < x \leq b \\
\dfrac{1}{2} & b < x \leq c \\
\dfrac{x+d-2c}{2(d-c)} & c < x \leq d \\
1 & x > d
\end{cases}
\tag{9}
$$

are *i.i.d.* random fuzzy variables with common average chance density:

$$
y(t) = \frac{e^{-at} - e^{-bt}}{2(b-a)t^2} + \frac{be^{-bt} - ae^{-at}}{2(b-a)t} + \frac{e^{-ct} - e^{-dt}}{2(d-c)t^2} + \frac{ce^{-ct} - de^{-dt}}{2(d-c)t}
\tag{10}
$$

Proof: Note that

$$
\Pr\{T(\lambda) \leq t\} = 1 - e^{\lambda t}
\tag{11}
$$

Therefore event $\{q : \Pr\{T(l(q)) \leq t\} \geq a\}$ is a fuzzy event and is equivalent to the fuzzy event $\{q : l(q) \geq -\ln(1-a)/t\}$. As a critical toward the derivation of the average chance distribution, it is necessary to calculate the credibility measure for fuzzy event $\{q : l(q) \geq -\ln(1-a)/t\}$, i.e., obtain the expression for

$$
Cr\{q : l(q) \geq -\ln(1-a)/t\}
\tag{12}
$$

Recall that for the credibilistic fuzzy variable, l, the credibility measure takes the form

$$
Cr\{q : l(q) \leq x\} = \begin{cases}
0 & x \leq a \\
\dfrac{x-a}{2(b-a)} & a < x \leq b \\
\dfrac{1}{2} & b < x \leq c \\
\dfrac{x+d-2c}{2(d-c)} & c < x \leq d \\
1 & x > d
\end{cases}
\tag{13}
$$

Accordingly, the range for integration with a can be determined as shown in Table 1. Recall that the expression of $x = -\ln(1-a)/t$ appears in Equations (12) and (13), which facilitates the link between intermediate variable α and average chance measure.

Table 1. Range analysis for α

x	a and credibility measure expression	
$-\Gamma < x \leq a$	Range for a	$0 \leq a \leq 1 - e^{-at}$
	$Cr\{l(q) \geq -\ln(1-a)/t\}$	1

$a < x \text{J} b$	Range for a	$1 - e^{-at} < a \text{ J } 1 - e^{-bt}$
	$\text{Cr} \{q : l (q)i - \ln(1- a)/t\}$	$1 - (x - a)/(2(b - a))$
$b < x \text{J} c$	Range for a	$1 - e^{-bt} < a \text{ J } 1 - e^{-ct}$
	$\text{Cr} \{q : l (q)i - \ln(1- a)/t\}$	0.5
$c < x \text{J} d$	Range for a	$1 - e^{-ct} < a \text{ J } 1 - e^{-dt}$
	$\text{Cr} \{q : l (q)i - \ln(1- a)/t\}$	$(d - x)/(2(d - c))$
$d < x < +\Gamma$	Range for a	$1 - e^{-dt} < a \text{ J } 1$
	$\text{Cr} \{q : l (q)i - \ln(1- a)/t\}$	0

The average chance distribution for the exponentially distributed random fuzzy lifetime is then derived by splitting the integration into five terms according to the range of α and the corresponding mathematical expression for the credibility measure $\text{Cr} \{q : l (q)i - \ln(1- a)/t\}$, which is detailed in Table 3. Then the exponential random fuzzy lifetime has an average chance distribution function:

$$Y(t) = \mathbf{T}_0^1 \text{Cr} \{q : l (q)i - \ln(1- a)/t\} da$$

$$= 1 + \frac{e^{-bt} - e^{-at}}{2(b - a)t} + \frac{e^{-dt} - e^{-ct}}{2(d - c)t}$$

(14)

and the average chance density is

$$y(t) = \frac{e^{-at} - e^{-bt}}{2(b - a)t^2} + \frac{be^{-bt} - ae^{-at}}{2(b - a)t}$$

$$+ \frac{e^{-ct} - e^{-dt}}{2(d - c)t^2} + \frac{ce^{-ct} - de^{-dt}}{2(d - c)t}$$

(15)

This concludes the proof.

Similar to the probabilistic reliability theory, we define a reliability function or survival function for a random fuzzy lifetime and accordingly name it as the average chance reliability function, which is defined accordingly as

$$\overline{Y}(t) = 1 - Y(t)$$

(16)

Then, for exponential random fuzzy lifetime, its average chance reliability function is

$$\overline{Y}(t) = \frac{e^{-at} - e^{-bt}}{2(b - a)t} + \frac{e^{-ct} - e^{-dt}}{2(d - c)t}$$

(17)

Theorem 5: The waiting time to the nth event, W_n, in a random fuzzy Poisson process $N = \{N(t), t i 0\}$ with fuzzy intensity l having a credibility distribution L has a average chance distribution function:

$$Y(t) = \mathbf{T}_0^1 \left\{ 1 - L \left[\frac{c_{2n,1- a}^2}{2t} \right] \right\} da$$

(18)

Proof: Note that the waiting time for a fixed l_0 follows a gamma density

$$f_{W_n}(t) = \frac{(l t)^{n-1}}{(n - 1)!} l e^{-l t}.$$

Further note that $2\lambda_0 W_n : \chi_{2n}^2$, therefore event

$$\left\{ q : \Pr\left\{ W_n(q) \leq t \right\} \geq \alpha \right\}$$

$$= \left\{ q : \Pr\left\{ 2\lambda W_n(q) \leq 2\lambda t \right\} \geq \alpha \right\}$$

$$= \left\{ q : \Pr\left\{ \chi_{2n}^2(q) \leq 2\lambda t \right\} \geq \alpha \right\} \tag{19}$$

$$= \left\{ q : \lambda(q) \geq \frac{\chi_{2n,1-\alpha}^2}{2t} \right\}$$

Hence the average chance distribution

$$\Upsilon_{W_n}(t)$$

$$= \int_0^1 Cr\left(q : \Pr\left\{ W_n(q) \leq t \right\} \geq \alpha \right) d\alpha \tag{20}$$

$$= \int_0^1 \left\{ 1 - \Lambda\left(\frac{\chi_{2n,1-\alpha}^2}{2t} \right) \right\} d\alpha$$

If we specify the form of the credibility distribution of Λ, then specific form of average chance distribution should be obtained.

3.3 Time-dependent random fuzzy Poisson processes

In reliability engineering and risk analysis, the non-stationary Poisson processes enjoy wide applications because the intensity function is time-dependent. It is expected that the mathematical treatments may be much more complicated since the fuzzy functional nature of intensity when the parameters are credibilistic fuzzy variables. For a concrete discussion purpose, we narrow our attention to a linear intensity function:

$$\lambda(t) = b_0 + b_1 t, \ b_0 > 0, \ b_1 > 0 \tag{21}$$

Further, we assume that b_0 and b_1 both have piecewise linear credibility distribution:

$$\Lambda_i(x) = \begin{cases} 0 & x < a_i \\ \dfrac{x - a_i}{2(b_i - a_i)} & a_i \leq x < b_i \\ \dfrac{x + c_i - 2b_i}{2(c_i - b_i)} & b_i \leq x < c_i \\ 1 & x \geq c_i \end{cases}, \ i = 0,1 \tag{22}$$

Then the integrated the intensity function (mean measure):

$$m(t) = b_0 t + b_1 t^2 \tag{23}$$

will have a credibility distribution:

$$
L_{m(t)}(y) = \begin{cases} 0 & y < a \\ \dfrac{y - a}{2(b - a)} & a \leq y < b \\ \dfrac{y + c - 2b}{2(c - b)} & b \leq y < c \\ 1 & y \geq c \end{cases}
$$

(24)

where

$$
\begin{aligned}
a &= a_0 t + a_1 t^2 \\
b &= b_0 t + b_1 t^2 \\
c &= c_0 t + c_1 t^2
\end{aligned}
$$

(25)

In general, the credibility distribution of the integrated intensity function $m(t)$, it is necessary to apply Zadeh's (1978) extension principle, denoted as $L_{m(t)}$, but for the piecewise linear credibility distribution case, the mathematical arguments are simpler.

Now let us derive the average chance distribution for the inter-arrival times.

$$
Y_T(t) = \int_0^1 Cr\left(q : \Pr\left\{T(q) \leq t\right\} \geq a\right) da
$$

(26)

Note that for the first arrival time,

$$
\left\{q : \Pr\left\{T_1(q) \leq t\right\} \geq a\right\}
$$

$$
= \left\{q : 1 - \exp\left\{-\int_0^t (b_0 + b_1 u) du\right\} \geq a\right\}
$$

(27)

$$
= \left\{q : 1 - e^{-m(t)} \geq a\right\}
$$

$$
= \left\{q : m(t) \geq -\ln(1 - a)\right\}
$$

Therefore, the average chance distribution for T_1, the first inter-arrival time, is

$$
\begin{aligned}
Y_{T_1}(t) \\
&= \int_0^1 Cr\left(q : \Pr\left\{T(q) \leq t\right\} \geq a\right) da \\
&= \int_0^1 Cr\left(q : m(t) \geq -\ln(1 - a)\right) da
\end{aligned}
$$

(28)

It is noticed that $y = -\ln(1 - a)$, therefore,

$$
Cr\left\{m(t) > y\right\} = \begin{cases} 1 & y < a \\ \dfrac{2b - 2 - y}{2(b - a)} & a \leq y < b \\ \dfrac{c - y}{2(c - b)} & b \leq y < c \\ 0 & y \geq c \end{cases}
$$

(29)

Table 2. Range analysis for α

y	a and credibility measure expression	
$-\Gamma < y \leq a$	Range for a	$0 \leq a \leq 1 - e^{-m(t)}$
	$Cr\{m(q)i - \ln(1-a)\}$	1
$a < y \leq b$	Range for a	$1 - e^{-at} < a \leq 1 - e^{-bt}$
	$Cr\{m(q)i - \ln(1-a)\}$	$(2b - a - y)/(2(b-a))$
$b < y \leq c$	Range for a	$1 - e^{-bt} < a \leq 1 - e^{-ct}$
	$Cr\{m(q)i - \ln(1-a)\}$	$(c - y)/(2(c-b))$
$c < y < +\Gamma$	Range for a	$1 - e^{-dt} < a \leq 1$
	$Cr\{m(q)i - \ln(1-a)\}$	0

Hence,

$$Y_{T_1}(t) = \int_0^1 Cr(q: m(t)i - \ln(1-a))da$$

$$= \int_0^{1-e^{-m(a)}} 1r\, da + \int_{1-e^{-m(a)}}^{1-e^{-m(b)}} \frac{2b - a + \ln(1-a)}{2(b-a)}da$$

$$+ \int_{1-e^{-m(b)}}^{1-e^{-m(c)}} \frac{c + \ln(1-a)}{2(b-a)}da + \int_{1-e^{-m(a)}}^{1} 0r\, da$$

$$= 1 - e^{-m(a)} + \frac{2b - a}{2(b-a)}\left(e^{-m(a)} - e^{-m(b)}\right) \tag{30}$$

$$+ \frac{1}{2(b-a)} \int_{1-e^{-m(a)}}^{1-e^{-m(b)}} \ln(1-a)da$$

$$+ \frac{c}{2(c-b)}\left(e^{-m(b)} - e^{-m(c)}\right)$$

$$+ \frac{1}{2(c-b)} \int_{1-e^{-m(b)}}^{1-e^{-m(c)}} \ln(1-a)da$$

Note that

$$\int \ln(1-a)da = (1-a) - (1-a)\ln(1-a) \tag{31}$$

Hence

$$\int_{1-e^{-m(a)}}^{1-e^{-m(b)}} \ln(1-a)da = \left(e^{-m(b)} - e^{-m(a)}\right) \tag{32}$$

$$- m(b)e^{-m(b)} + m(a)e^{-m(a)}$$

and

$$\int_{1-e^{-m(b)}}^{1-e^{-m(c)}} \ln(1-a)da = \left(e^{-m(c)} - e^{-m(b)}\right)$$
$$- m(c)e^{-m(b)} + m(b)e^{-m(b)} \tag{33}$$

Combine above arguments, it is established that:

$$Y_{T_1}(t)=$$
$$1 - e^{-m(a)} + \frac{2b - a - 1}{2(b-a)}\left(e^{-m(a)} - e^{-m(b)}\right)$$
$$+ \frac{1}{2(b-a)}\left(-m(b)e^{-m(b)} + m(a)e^{-m(a)}\right)$$
$$+ \frac{c-1}{2(c-b)}\left(e^{-m(b)} - e^{-m(c)}\right) \tag{34}$$
$$+ \frac{1}{2(c-b)}\left(-m(c)e^{-m(c)} + m(b)e^{-m(b)}\right)$$

Next let us derive the ith inter-arrival time. Recall that conditioning on the $(i-1)^{st}$ occurrence time w_{i-1}, the mean measure is

$$m(t \mid w_{i-1}) = b_0(t - w_{i-1}) + b_1\left(t^2 - w_{i-1}^2\right) \tag{35}$$

Accordingly, the credibility distribution for $m(t \mid w_{i-1})$ is

$$L_{m(t \mid w_{i-1})}(y) = \begin{cases} 0 & y < a \\ \dfrac{y-a}{2(b-a)} & a \leq y < b \\ \dfrac{y+c-2b}{2(c-b)} & b \leq y < c \\ 1 & y \geq c \end{cases} \tag{36}$$

where

$$a = a_0(t - w_{i-1}) + a_1\left(t^2 - w_{i-1}^2\right)$$
$$b = b_0(t - w_{i-1}) + b_1\left(t^2 - w_{i-1}^2\right) \tag{37}$$
$$c = c_0(t - w_{i-1}) + c_1\left(t^2 - w_{i-1}^2\right)$$

Thus, the average chance distribution for the ith inter-arrival time is:

$$Y_{T_i}(t) =$$

$$1 - e^{-m(a|w_{i-1})} + \frac{2b - a - 1}{2(b-a)}\left(e^{-m(a|w_{i-1})} - e^{-m(b|w_{i-1})}\right)$$

$$+ \frac{1}{2(b-a)}\left(-m(b|w_{i-1})e^{-m(b|w_{i-1})} + m(a|w_{i-1})e^{-m(a|w_{i-1})}\right) \tag{38}$$

$$+ \frac{c-1}{2(c-b)}\left(e^{-m(b|w_{i-1})} - e^{-m(c|w_{i-1})}\right)$$

$$+ \frac{1}{2(c-b)}\left(-m(c|w_{i-1})e^{-m(c|w_{i-1})} + m(b|w_{i-1})e^{-m(b|w_{i-1})}\right)$$

It is necessary to emphasize that in the expression of the average chance distribution of the inter-arrivals (either Equation (34) for the first arrival, or Equation (38) for the ith arrival) the time t factor is containing in the parameters (a,b,c) as shown in Equation (25) for the first arrival and Equation (37) for the ith arrival. Also, the functions for parameters (a,b,c) are changed for the successive arrivals as indicated in Equation (37).

4   A PARAMETER ESTIMATION SCHEME

The parameter estimation is in nature an estimation problem of credibility distribution from fuzzy observations. Guo and Guo (2009) recently proposed a maximally compatible random variable to a credibilistic fuzzy variable and thus the fuzzy estimation problem is converted into estimating the distribution function of the maximally compatible random variable. The following scheme is for estimating the piecewise linear credibility distribution.

Definition 8: Let X be a random variable defined in $(R, B(R))$ such that

$$\mu^c = Cr \circ \xi^{-1} = \mu = P \circ X^{-1} \tag{39}$$

Then X is called a maximally compatible to fuzzy variable ξ.

In other words, random variable X can take all the possible real-values the fuzzy variable ξ may take with and the distribution of X, $F_X(r)$ equals the credibility distribution of x, $L_x(r)$ for all $r \in R$.

It is aware that the induced measure $\mu^c = Cr \circ \xi^{-1}$ and measure $\mu = P \circ X^{-1}$ are defined on the same measurable space $(R, B(R))$. Furthermore, we notice that the pre-image $\xi^{-1}(B) \in P(\Theta)$, but, the pre-image $X^{-1}(B) \in A(\Theta) \subset P(\Theta)$, which implies that for the same Borel set $B \in B(R)$, the pre-images under fuzzy variable ξ and random variable are not the same. It is expected that

$$\{q O Q : X(q) J \ r\} H \{q O Q : x(q) J \ r\} \tag{40}$$

but

$$Pr\{q O Q : X(q) J \ r\}$$
$$= Cr\{q O Q : x(q) J \ r\} \tag{41}$$

The statistical estimation scheme for parameters (a,b,c) of the credibility distribution based on fuzzy observations $\{x_1, x_2, \cdots, x_n\}$ can be stated as:

Estimation Scheme 1:

Step 1: Rank fuzzy observations $\{x_1, x_2, \cdots, x_n\}$ to obtain "order" statistics $\{x_{(1)}, x_{(2)}, \cdots, x_{(n)}\}$ in ascending order;

Step 2: Set $\hat{a} = x_{(1)}$ and $\hat{c} = x_{(n)}$;

Step 3: Set a tentative estimator for b,

$$\hat{b}_e = \frac{4\overline{x}_n - x_{(1)} - x_{(n)}}{2} \tag{42}$$

where

$$\overline{x}_n = \frac{1}{n}\sum_{i=1}^{n} x_i \tag{43}$$

Step 4: Identify $x_{(i_0)}$ from $\left\{x_{(1)}, x_{(2)}, \cdots, x_{(n)}\right\}$ such that $x_{(i_0)} \text{J} \ \hat{b}_e < x_{(i_1)}$ and $1 < i_0 < i_1$, then we may see $\left\{x_{(1)}, x_{(2)}, \cdots, x_{(i_0)}\right\}$ as a set of order statistics from uniform $[a,b]$. Hence the "sufficient" statistic for parameter b is $x_{(i_0)}$.

Then $\left(\hat{a}, \hat{b}, \hat{c}\right) = \left(x_{(1)}, x_{(i_0)}, x_{(n)}\right)$ is the parameter estimator for the piecewise linear credibility distribution.

$$\hat{\Lambda}(x) = \begin{cases} 0 & x < \hat{a} \\ \dfrac{x - \hat{a}}{2\left(\hat{b} - \hat{a}\right)} & \hat{a} \leq x < \hat{b} \\ \dfrac{x + \hat{c} - 2\hat{b}}{2\left(\hat{c} - \hat{b}\right)} & \hat{b} \leq x < \hat{c} \\ 1 & x \geq \hat{c} \end{cases} \tag{44}$$

The next issue is how to extract the information on intensity rate λ in stationary random fuzzy Poisson process.

It is noticed that for probabilistic Poisson process case, the interpretation of intensity λ is the occurrence rate in unit time. Based on such an observation, therefore, for any individual value λ_0 the fuzzy intensity may take, it results in a probabilistic Poisson process. Sample this Poisson process until n_0 events and record the total waiting time w_{n_0}, then $\hat{\lambda}_0 = n_0/w_{n_0}$ is an estimate of intensity λ_0. Repeat the sampling procedure from the random fuzzy Poisson process as many times as possible, say, m times, then the intensity "observation" sequence is

$$\left\{\hat{\lambda}_1, \hat{\lambda}_2, \text{L}, \hat{\lambda}_m\right\} = \left\{\frac{n_0^1}{w_{n_0}^1}, \frac{n_0^2}{w_{n_0}^2}, \text{L}, \frac{n_0^m}{w_{n_0}^m}\right\} \tag{45}$$

Apply the Estimation Scheme 1 to the estimated rate observations $\left\{\hat{\lambda}_1, \hat{\lambda}_2, \text{L}, \hat{\lambda}_m\right\}$ the piecewise linear credibility distribution shown in Equation (44).

For the non-stationary random fuzzy Poisson process, the mean measure involves two linear piecewise credibility distributions for fuzzy parameter s b_0 and b_1 respectively.

The scheme can state as follows:

Step 1: Sampling procedure from the random fuzzy Poisson process $N = \left\{N_t, t \text{ i } 0\right\}$ m times. Let the ith n_0 events the waiting time are $\left\{w_1^i, w_2^i, \text{L}, w_{n_0}^i\right\}$.

Step 2: For the ith sample, perform the maximum likelihood estimation and obtain the parameter pair $\left(\hat{b}_0^i, \hat{b}_1^i\right)$ which is regarded as the fuzzy parameters taking values. Repeat the estimation process until all m MLE pairs $\left\{\left(\hat{b}_0^i, \hat{b}_1^i\right), i = 1, 2, \text{L}, m\right\}$ are obtained.

Step 3: Applying the Estimation Scheme 1 to fuzzy sequences $\{\hat{b}_0^1, \hat{b}_0^2, L, \hat{b}_0^m\}$ and $\{\hat{b}_1^1, \hat{b}_1^2, L, \hat{b}_1^m\}$ respectively, the parameters $(\hat{a}_0, \hat{b}_0, \hat{c}_0)$ and $(\hat{a}_1, \hat{b}_1, \hat{c}_1)$ define the two piecewise linear credibility distributions for b_0 and b_1 respectively.

5 A SIMULATION SCHEME

Simulation of a random fuzzy Poisson process is intrinsically two-stage procedure: a fuzzy parameter simulation for generating realizations $\{l_1, l_2, L, l_n\}$ from a piecewise linear credibility distribution function L and a waiting times sequence:

$$W_n = \sum_{i=1}^{n} T_i \tag{46}$$

where T_1, T_2, L, T_n are i.i.d. exponential with common probability density function $l \, e^{-lt}$.

As to the fuzzy parameter simulation, we utilize the maximally compatible random variable to a fuzzy variable concept and the inverse transformation of the probability distribution function approach for generating fuzzy variable realizations. An algorithm is stated as follows:

Fuzzy simulation scheme 1:

Step 1: Simulating uniform random variable uniform[0,1], and denote the simple random sample as $\{u_1, u_2, \cdots, u_n\}$;

Step 2: Set $\Lambda(x_i) = u_i$, $(i = 1, 2, \cdots, n)$;

Step 3: Set x_i, $(i = 1, 2, \cdots, n)$:

$$x_i = \begin{cases} a + 2(b - a)u_i & \text{if } 0 \le u_i \le 0.5 \\ 2b - c + 2(c - b)u_i & \text{if } 0.5 \le u_i \le 1 \end{cases} \tag{47}$$

Then $\{x_1, x_2, \cdots, x_n\}$ is a sample from the fuzzy variable ξ with a piecewise linear credibility distribution Λ.

Next we state a random fuzzy Poisson process simulation scheme.

Simulation scheme 2: Simulating a stationary random fuzzy Poisson process.

Step 1: Simulate a sequence of uniform (0,1), denoted as $\{u_1, u_2, L, u_n\}$,

Step 2: $\{t_1, t_2, L, t_n\}$

$$t_i = -\frac{1}{b}\ln(1 - u_i), \ i = 1, 2, L, n \tag{48}$$

are the exponentially distributed random lifetime.

Step 3: in term of Fuzzy Simulation Scheme 1, Fuzzy variable sample $\{x_1, x_2, \cdots, x_n\}$ is obtained.

Step 4: $\{T_1, T_2, L, T_n\}$

$$T_i = -\frac{1}{z_i}\ln(1 - u_i), \ i = 1, 2, L, n \tag{49}$$

which construct a stationary random fuzzy Poisson process $N = \{N_t, t \ge 0\}$.

As to the time-dependent random fuzzy Poisson process, we state an algorithm to illustrate the idea. For example, we simulate a random fuzzy Poisson process with power law intensity function having a fuzzy scale parameter.

Simulation scheme 3: Simulating a time-depenedent random fuzzy Poisson waiting times $\{W_i, i = 1, 2, L, n\}$, which forms a power law process. Note that the conditional Weibull distribution

$$F(t\,|\,x) = 1 - \exp\left\{-\left[\left(\frac{t+x}{h}\right)^b - \left(\frac{x}{h}\right)^b\right]\right\} \tag{50}$$

Then, **Step 1:** For given sample from Uniform(0,1), $\{u_1, u_2, \mathrm{L}, u_n\}$,

$$W_i = h\left\{\ln\left[e^{\left(\frac{W_{n-1}}{h}\right)} - \ln(1 - u_i)\right]\right\}^{1/b}, \quad i = 1, 2, \mathrm{L}, n \tag{51}$$

are waiting times in the random fuzzy Poisson process with power law.

Step 2: As to the fuzzy scale parameter, h, in term of *Fuzzy Simulation Scheme 1*, $\{x_1, x_2, \cdots, x_n\}$ is the sample from the fuzzy scale parameter Finally, let

$$W_i = x_i\left\{\ln\left[e^{\left(\frac{W_{n-1}}{x_i}\right)} - \ln(1 - u_i)\right]\right\}^{1/b}, \quad i = 1, 2, \mathrm{L}, n \tag{52}$$

will generate random fuzzy waiting times $\{W_1, W_2, \mathrm{L}, W_n\}$ with fuzzy scale parameter h, which construct a random fuzzy Poisson process with power law intensity having a fuzzy scale parameter h.

6 CONCLUSION

In this paper, we give a systematic treatment of random fuzzy Poisson processes not only from the stationary one and then non-stationary one, but also a parameter estimation scheme as well as a simulation scheme is proposed. In this way, the foundation for the random fuzzy Poisson processes is formed although in its infant stage, particularly, the time-dependent random fuzzy Poisson process. The applications to reliability engineering fields and the risk analysis now can extend from random uncertainty only cases to randomness and fuzziness co-existence cases. It is expecting that this development will help the reliability and risk analysis researchers as well as reliability analysts and engineers.

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