On Recent Generalizations of the Weibull Distribution

Hoang Pham, *Fellow, IEEE*, and Chin-Diew Lai

***Abstract*—This short communication first offers a clarification to a claim by Nadarajah & Kotz. We then present a short summary (by no means exhaustive) of some well-known, recent generations of Weibull-related lifetime models for quick information. A brief discussion on the properties of this general class is also given. Some future research directions on this topic are also discussed.**

***Index Terms*—Bathtub shape, failure rate function, Weibull distribution.**

NOTATION

T	Lifetime random variable
$f(t)$	Probability density function (pdf) of T
$F(t)$	Cumulative distribution function (cdf)
$h(t)$	hazard rate function
$H(t)$	Cumulative failure rate function $[H(t) = \int_0^t h(x)dx]$
$R(t)$	Reliability function $[= 1 - F(t)]$
$\mu(t)$	Mean residual life defined by $E(T - t \mid T > t)$
μ	Mean lifetime

ACRONYM[1]

IFR	Increasing failure rate
DFR	Decreasing failure rate
BT	Bathtub
MBT	Modified bathtub
UBT	Upside-down bathtub
MRL	Mean residual life
WPP	Weibull probability plot

I. INTRODUCTION

IN the context of reliability modeling, some well-known interrelationships between the various quantities such as pdf, cdf, failure rate function, cumulative failure rate function, and reliability function, for a continuous lifetime T, can be summarized as

$$h(t) = \frac{f(t)}{1 - F(t)} = \frac{f(t)}{R(t)} \tag{1}$$

$$H(t) = \int_0^t h(x)dx \tag{2}$$

$$R(t) = e^{-H(t)} \tag{3}$$

Note that all the cumulative failure rate functions must satisfy the following conditions:

i. $H(t)$ is nondecreasing for all $t \geq 0$
ii. $H(0) = 0$
iii. $\lim_{t\to\infty} H(t) = \infty$

Thus, knowing one of the three quantities, one can easily obtain the other two. In this short communication, we shall see how (3) facilitates the construct of Weibull-type lifetime distributions. The bathtub-shaped failure rate function plays an important role in reliability applications, such as human life, and electronic devices. Many authors have proposed new distributions based on the traditional Weibull distribution function. Nadarajah & Kotz [20] recently made the point that the proposed distributions, published in reliability engineering journals, are either not new, or arise from a representation suggested by Gurvich *et al.* [8]. They feel that Gurvich *et al.* [8] were the first to present a class of distributions generalizing the traditional Weibull distribution, and their work needs to be recognized by the readers of reliability journals.

This short communication first offers a clarification that the claim by Nadarajah & Kotz [20] is inaccurate because (8) in [8] (equivalent to (1) in [20]) is exactly one of many common fundamental interrelationships in reliability concepts (see [1], page 12, (2.1)]. Thus, the contribution by Gurvich *et al.* [8] has no direct relevance in reliability engineering. We then present a short summary (by no means exhaustive) of some well-known related Weibull distributions with two or more parameters, and their characteristics along with some recent distribution models for quick information. Some future research directions on this topic are also discussed.

II. CUMULATIVE FAILURE FUNCTION H AND GURVICH *ET AL.*'S G

Equation (3) above describes the relationship between the reliability function, and cumulative failure rate function; it can be found in, for example, Barlow & Proschan [1], page 12. Depending on how elaborate a lifetime distribution one wishes to obtain, one can simply use H(t) (or the failure rate function h(t)) to yield more complex, or less complex analytic function for R(t). In the past decades, many authors [2]–[4], [6]–[10], [13], [15]–[18], [20]–[22], [26]–[35] have proposed, or studied lifetime distributions with various bathtub-shaped failure rates.

In the context of modeling the random strength of brittle materials, Gurvich *et al.* [8] considered a class of distributions generalizing the traditional Weibull model:

$$R(t) = \exp\{-\alpha G(t)\}, \alpha > 0 \tag{4}$$

where $G(t)$ is a monotonically increasing function of t. Comparing (3) to (4), we immediately note that $\alpha G(t) = H(t)$. This implies that the class represented by (4) is a well-known general result in reliability literature.

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H. Pham is with the Department of Industrial and Systems Engineering, Rutgers University, NJ, USA.

C.-D. Lai is with Massey University, Auckland, New Zealand.

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[1]The singular and plural of an acronym are always spelled the same.

Nadarajah & Kotz [20] recently claim that "Because of the obvious flexibility of (1), we feel that it is important that the work of Gurvich *et al.* [8] is recognized by the general audience of reliability journals."[2] This claim is therefore inaccurate in view of our discussion above.

III. Recent Development in Weibull Related Models

A life distribution can be classified according to the shape of its $h(t)$, or its $\mu(t)$. For convenience sake, the life classes are often given by their abbreviations IFR, DFR, BT, MBT, etc. We say that F is IFR (DFR) if $h(t)$ is nondecreasing (nonincreasing) in t; BT (UBT) if $h(t)$ has a bathtub (upside-down bathtub) shape; MBT (modified bathtub) if $h(t)$ is first increasing, then followed by a bathtub curve. For more details concerning these and other aging classes, see Lai & Xie ([12], Section 2.4).

Despite its popularity, and wide applications, the traditional 2- or 3-parameter Weibull distribution is unable to capture the behavior of a lifetime data set that has a non-monotonic failure rate function $h(t)$. For this reason, many aging distributions were proposed to overcome this deficiency. Taking the advantage of the relationship given in (3), or (4), one can construct a generalized Weibull distribution by selecting an easily differentiable function $H(t)$. We note that $H'(t) = h(t)$. With a suitable choice of $H(t)$, and its parameter, we can obtain a bathtub shaped failure rate distribution. In many practical applications, $h(t)$ is initially decreasing, followed by a period of approximately constant hazard, and ultimately increasing because of the eventual positive aging effect. It has widely been believed that many products, particularly electronic items such as silicon integrated circuits, exhibit a bathtub shaped failure rate function. This belief is supported by much experience, and extensive data collection in many industries.

For most of the existing generalized Weibull distributions in reliability literature, the function $H(t)$ is quite simple so that both $f(t)$, and $h(t)$ have a simple form. In general, the moments of Weibull related models are often difficult to obtain, but their quantiles are not hard to find. In particular, the median can be easily obtained. Also, one can usually construct a generalized probability plot similar to the well-known Weibull probability plot (WPP). A WPP plot allows the model builder to determine whether one or more of the Weibull models are suitable for modeling a given data set. It also provides some initial estimates of the parameters concerned. See Section 11.3 of Murthy *et al.* [19] for a discussion.

Table I offers a brief but by no means exhaustive summary of common well-known lifetime distributions with two or more parameters, and their characteristics, along with some recent distributions appearing in the reliability engineering literature. Table II summarizes some other generalized Weibull models that do not have a simple form for $R(t)$, but nevertheless is related to the Weibull one way or another. Without loss of generality, we may classify a generalized Weibull distribution as one of the following:

1) Involve with one single Weibull: a) linear transformation; b) power transformation; c) log transformation; d) nonlinear transformation; and e) Inverse transformation.
2) By adding one or more parameters: a) adding a location parameter; b) adding a shape parameter; c) adding a scale parameter; d) adding a location and a shape parameter; e) adding a parameter that serves as a hybrid of a shape and a scale parameter; f) adding two location parameters, and g) truncation.
3) Power transformation of cumulative distribution, or survival function.
4) As a modification of some generalized gamma distribution.
5) Mixtures: a) mixtures of two or more Weibull distributions; b) mixture of a singular distribution with a Weibull; and c) mixtures of two generalized Weibull distributions.
6) Construct a new failure rate function $h(t)$ as a simple generalization of the failure rate function of Weibull. Then a generalized Weibull can be obtained:

$$R(t) = \exp\left\{-H(t)\right\}, \quad \alpha > 0 \text{ where } H(t) = \int_0^t h(x)dx.$$

7) Involving two or more Weibull distributions: a) finite mixtures; b) n-fold competing risk (equivalent to independent components being arranged in a series structure); c) n-fold multiplicative models; and d) n-fold sectional models.
8) Weibull with varying parameters. The parameter(s) are either a function of time t, or some other variables such as stress level, or are random variables.

IV. Recent Weibull Models

Most of the generalizations or modifications of Weibull distributions listed in the tables have been discussed in Murthy *et al.* [19]. We now briefly discuss some of the recent Weibull studies that appeared since 2004.

i) Based on (4), Nadarajah & Kotz [20] proposed a generalization that contains the model of Xie *et al.* [34], with Chen [5] as a particular case

$$R(t) = \exp\left\{-at^b\left(e^{ct^d} - 1\right)\right\}, a, d > 0; b, c \geq 0; t \geq 0.$$

ii) Muralidharan & Lathika [18] considered a lifetime phenomenon with instantaneous, or early failures. They showed that such situations can be modeled by mixing a Weibull distribution with a singular distribution at zero (or at $t = \delta$), thus resulting in a generalized Weibull model. Let F be either a two-parameter, or a three-parameter Weibull distribution; F_{M} be the cdf of the resulting mixture, and $R(t) = 1 - F_M(t)$, then the distribution of the instantaneous failure Weibull model can be expressed as

$$F_M(t) = \begin{cases} 1-\alpha & t = 0 \\ 1-\alpha+\alpha F(t) & t > 0 \end{cases}$$

[2]In the original quote, the reference is numbered as [1], but we renumbered it to match our references for your convenience.

TABLE I
RELIABILITY FUNCTIONS FOR SOME COMMON GENERALIZED WEIBULL DISTRIBUTIONS

Author	Reliability function	Characteristics
Gompertz (1825) [7]	$R(t)=\exp\left\{\frac{\theta}{\alpha}\left(1-e^{\alpha t}\right)\right\}$, $\theta>0, -\infty<\alpha<\infty;\ t\ge 0$.	IFR if $\alpha>0$ DFR if $\alpha<0$
Weibull (1951) [30]	$R(t)=\exp\left(-\lambda t^{\alpha}\right), \lambda,\alpha>0;\ t\ge 0$.	Exponential. if $\alpha=1$ IFR if $\alpha>1$ DFR if $\alpha<1$
Smith & Bain (1975) [28](Exponential power model)	$R(t)=\exp\left\{1-e^{(\lambda t)^{\alpha}}\right\};\alpha,\lambda>0; t\ge 0$.	BT if $0<\alpha<1$ Special cases of Chen's model
Xie & Lai (1995) [33]	$R(t)=\exp\left\{-(t/\beta_1)^{\alpha_1}-(t/\beta_2)^{\alpha_2}\right\}$ $\alpha_1,\alpha_2,\beta_1,\beta_2>0;\ t\ge 0$.	IFR if $\alpha_1,\alpha_2>1$ DFR if $\alpha_1,\alpha_2<1$ BT if $\alpha_1<1,\ \alpha_2>1$
Chen (2000) [5]	$R(t)=\exp\left\{-\lambda\left[e^{t^{\beta}}-1\right]\right\}, \lambda,\beta>0; t\ge 0$	BT if $\beta<1$ IFR if $\beta\ge 1$ Exponential power if $\lambda=1$
Pham (2002) [26]	$R(t)=\exp\left\{1-a^{t^{\alpha}}\right\};\alpha,a>0;\ t\ge 0$.	DFR for $t\le t_0$ IFR for $t\ge t_0$ where $t_0=\left(\frac{1-\alpha}{\alpha\ln a}\right)^{\frac{1}{\alpha}}$
Xie, Tang & Goh (2002) [34]	$R(t)=\exp\left\{\lambda\beta\left[1-e^{(t/\beta)^{\alpha}}\right]\right\}$ $\alpha,\beta,\lambda>0;\ t\ge 0$.	Chen's model if $\beta=1$ IFR if $\alpha\ge 1$ BT if $0<\alpha<1$
Lai, Xie & Murthy (2003) [13] Gurvich, Dibenedetto & Rande (1997) [8]	$R(t)=\exp\left\{-at^{\alpha}e^{\lambda t}\right\}$ $\lambda\ge 0,\ \alpha,a>0;\ t\ge 0$.	Weibull if $\lambda=0$: Exponential if $\beta=0, \alpha=1$ IFR if $\alpha\ge 1$ BT if $0<\alpha<1$
Nadarajah & Kotz (2005) [20] †	$R(t)=\exp\left\{-at^{b}\left(e^{ct^{d}}-1\right)\right\}$ $a,d>0; b,c\ge 0;\ t\ge 0$.	Xie, Tang & Goh's model if $b=0$ Chen's model if $b=0$, $c=1$
Bebbington, Lai & Zitikis (2006) [3]]	$R(t)=\exp\left\{-\left(e^{\alpha t\ \beta t}\right)\right\};\alpha,\beta>0; t\ge 0$.	IFR iff $\alpha\beta>(27/64)$ MBT if $\alpha\beta<(27/64)$

† Contrary to what was stated in Nadarajah & Kotz [20], this model does not contain the model given by Lai *et al.* [13]

Similarly, the distribution F_{M} with early failure can be obtained by mixing the two distributions D, and F as

$$F_M(t)=(1-\alpha)D(t)+\alpha F(t)$$

where $D(t)$ is a discrete (singular) distribution with mass at $t=\delta$. Here, the known δ is assumed to be small, and specified in advance.

iii) Nikulin & Haghighi [24] proposed a generalized power Weibull distribution with three parameters, and its reliability function

$$R(t)=\exp\left\{1-\left(1+(t/\beta)^{\alpha}\right)^{\theta}\right\},\quad t\ge 0;\alpha,\beta>0,\theta\ge 0. \tag{5}$$

Note that when $\theta=1$, (5) reduces to a two-parameter Weibull; when $\theta=1$, and $\alpha=1$, it reduces to the exponential. They also showed that $h(t)$ is i) IFR if either $\alpha>1$, and $\alpha>1/\theta$; or $\alpha=1$, and $\theta>1$; ii) DFR if either $0<\alpha<1$, and $\alpha<1/\theta$; or $\alpha\theta=1$, and $0<\alpha<1$; iii) BT if $0<1/\theta<\alpha<1$; and iv) UBT if $1/\theta>\alpha>1$.

iv) Bebbington *et al.* [3] obtained a generalization of Weibull having a simple, yet flexible cumulative failure rate H:

$$R(t)=\exp\left\{-\left(e^{\alpha t-\beta/t}\right)\right\};\alpha,\beta>0;t\ge 0. \tag{6}$$

Equation (6) reduces to a standard Weibull when $\beta=0$, and let $\alpha=\log\lambda$.It was shown that F is IFR iff $\alpha\beta<(27/64)$, and MBT if $\alpha\beta\le(27/64)$.

V. CONCLUDING REMARKS

Many generalized Weibull models have been proposed in reliability literature through the fundamental relationship between the reliability function $R(t)$, and its corresponding cumulative failure rate function $H(t)$. In this note, we summarize some commonly known models, and also discuss their general properties with a hope to provide practitioners a quick overview of the most recent developments in reliability concerning the Weibull distribution. Most generalizations of the Weibull distribution stemmed from a desire to provide a better fitting of certain data sets than the traditional two- or three- parameter Weibull. One would expect many more such generalizations, modifications, or extensions to appear in years to come. Given a data set, a researcher has an onerous task to select an 'optimal' model among many possible Weibull related models.

In general, there are three steps involving the empirical modeling of data, including Weibull, such as model selection, estimation of model parameters, and model validation. On parameter estimation, the number of parameters could be pivotal; and

TABLE II
RELIABILITY FUNCTIONS FOR SOME PARAMETRIC DISTRIBUTIONS THAT CAN BE CONSIDERED AS GENERALIZED WEIBULL DISTRIBUTIONS

Author	Reliability function	Characteristics
White (1969) [16]‡ (log Weibull)	$R(t)=\exp\left\{-\exp\left(\frac{t-a}{b}\right)\right\},\ -\infty<t<\infty;$ $-\infty<a<\infty,\ b>0$	$h(t)$ defined over $(-\infty,\infty)$
Hjorth (1980) [9]	$R(t)=\frac{\exp\left(-\delta t^2/2\right)}{(1+\beta t)^{\theta/\beta}},\ \delta,\beta,\theta>0;t\geq 0.$	BT shape if $0<\delta<\theta\beta$
Slymen & Lachenbruch (1984) [29]	$R(t)=\exp\left\{-\exp\left[\alpha+\frac{\beta(t^{\theta}-t^{-\theta})}{2\theta}\right]\right\},t\geq 0$	BT if $2\{(\theta+1)t-\theta-2-(\theta-1)t\theta-2\}$ $\times(t\theta-1-t-\theta-1)-2$ bounded
Phani (1987) [27]	$R(t)=\exp\left\{-\lambda\frac{(t-a)^{\beta_1}}{(b-t)^{\beta_2}}\right\}$ $\lambda>0,\beta's>0,\ 0\leq a\leq t\leq b<\infty$	Kies (1958)[9] if $\beta_1=\beta_2=\beta$ IFR if $\beta\geq 1$ BT if $0<\beta<1$
Mudholkar & Srivastava (1993) [16] (Exponentiated Weibull)	$R(t)=1-\left[1-\exp\left(-t/\beta\right)^{\alpha}\right]^{\theta},t\geq 0;$ $\alpha,\beta>0,\theta\geq 0$	Weibull if $\theta=1$ Exponential if $\alpha=\theta=1$ DFR if $\alpha,\theta<1$ IFR if $\alpha,\theta>1$ BT of IFR if $\alpha>1,\theta<1$: UBT of DFR if $\alpha<1,\theta>1$:
Mudholkar, Srivastava & Kollia (1996) [17]	$R(t)=1-\left[1-\left(1-\lambda\left(\frac{t}{\beta}\right)^{\alpha}\right)^{1/\lambda}\right],$ $\alpha,\beta>0;\ t\geq 0$	BT for $\alpha<1,\ \lambda>0$ IFR for $\alpha\geq 1,\ \lambda\geq 0$ DFR for $\alpha\leq 1,\ \lambda\geq 0$ UBT for $\alpha>1,\ \lambda>0$ Exponential for $\alpha=1,\ \lambda=0$
Marshall Olkin (1997) [15]	$R(t)=\frac{\nu\exp\left[-(t/\beta)^{\alpha}\right]}{1-(1-\nu)\exp\left[-(t/\beta)^{\alpha}\right]}$ $\alpha,\beta,\nu>0;\ t\geq 0.$	IFR if $\nu\geq 1,\ \alpha\geq 1$ DFR if $\nu\geq 1,\ \alpha\leq 1$ MBT $\alpha=2,\nu=0.1$ or 0.05
Inverse Weibull (Jiang et al. 2001) [10]	$R(t)=1-\exp\left(-(\beta/t)^{\alpha}\right),\alpha,\beta>0;t\geq 0.$	UBT
Nikulin & Haghighi (2006) [24]	$R(t)=\exp\left\{1-\left(1+\left(t/\beta\right)^{\alpha}\right)^{\theta}\right\},t\geq 0;\alpha,\beta>0,\theta\geq 0$	Can achieve various shapes, See Section 6 below.

‡ log Weibull is not a lifetime distribution.

how easily the estimates of these parameters can be found is also an important factor. The principle of parsimony may apply in this case. An overcomplicated Weibull model often diminishes the possibility of interpreting the parameters. Generally speaking, a Weibull model that has more than three parameters is undesirable, with the exception of mixtures of two modified Weibull distributions, which typically require 5–6 parameters including the mixing proportion. This is because the parameters in a mixture distribution are 'separated' owing to the additive nature of both the distribution function, and the density function. Thus mathematical operations on the failure rate can be obtained relatively easily, and therefore, numerical estimates of parameters become less prohibitive. Although the mixture of two 'pure' Weibull distributions cannot yield a BT distribution, its failure rate can achieve one of eight different types including IFR, DFR, MBT and 'roller-coaster' shaped. See, for example, Section 2.8.4 of Lai & Xie [12]. Our more recent search indicates that a mixture of two (non-identical) modified Weibull distributions has a lot to offer.

The Weibull, and related models have been used in many applications, and for solving a variety of problems from many disciplines. Table 1.1 of Murthy *et al.* [19] gives a small sample of such applications. Although the Weibull distribution is primarily used for modeling product failures in reliability engineering, it is also sometime used to model the human aging process (Eakin *et al.* [6]). Recently, Bebbington *et al.* [4] have modeled human mortality using mixtures of two different modified Weibull distributions. We feel that various mixtures of two modified Weibull distributions could have a wide range of possible applications, and it is worth further study.

Another important measure of reliability is the mean residual life $\mu(t)$, which is the expected remaining life beyond the present age t. Mathematically, it is defined as $\mu(t)=[\int_t^{\infty}R(x)dx]/R(t)$. The mean residual life can also be related to the failure rate $h(t)$ through $\mu'(t)=\mu(t)h(t)-1$. In industrial reliability studies of repair, replacement, and other maintenance strategies, the mean residual life function may be proven to be more relevant than the failure rate function. Indeed, if the goal is to improve the average system lifetime, then the mean residual life is the relevant measure. The function $h(t)$ relates only to the risk of immediate failure, whereas $\mu(t)$ summaries the entire residual life distribution.

For generalized Weibull distributions, the mean residual life $\mu(t)$ is generally difficult to obtained explicitly. However, for the exponentiated Weibull (see Table II) of Mudholkar & Srivastava [16], its $\mu(t)$ is derived by Nassar & Eissa [22] for $\theta = 2$. Although the explicit expressions for these life distributions are complex in general, their shapes can be determined in view of their relationships with the failure rate functions (see [12] Chapter 4). For example, suppose $h(t)$ has a BT shape, then $\mu(t)$ has an UBT shape if $h(0) > 1/\mu$, and is decreasing if $h(0) \leq 1/\mu$. Bebbington *et al.* [2] recently proposed using the curvature of the function $h(t)$, and $\mu(t)$ to identify a useful period of a bathtub-shaped life distribution. A generalized Weibull model is used to illustrate their procedure. Further analysis on the relationship between the mean residual life, and the failure rate functions for various generalizations of Weibull distributions is worth exploring.

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Hoang Pham received the Ph.D. degree in industrial engineering from the State University of New York at Buffalo. He is currently a Professor, and the Chairman of the Department of Industrial and Systems Engineering at Rutgers University. He is the author of 4 books, has published more than 90 journal articles, and has edited more than 10 books. He is editor-in-chief of the *International Journal of Reliability, Quality and Safety Engineering*, associate editor of the IEEE TRANSACTIONS ON SYSTEMS, MAN AND CYBERNETICS, and an editorial board member of a dozen journals. He is a fellow of the IEEE.

Chin-Diew Lai received the Ph.D. degree in statistics from Victoria University of Wellington in 1975. He has authored or co-authored over 90 journal articles, and book chapters. His research interests include statistical reliability, statistical quality control, and probability distributions.