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Interval optimization of dynamic response for structures with interval parameters

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Abstract

This paper presents an interval optimization method for the dynamic response of structures with interval parameters. The matrices of structures with interval parameters are given. Combining the interval extension of function with the perturbation theory of dynamic response, the method for interval dynamic response analysis is derived. The interval optimization problem is transformed into a corresponding deterministic one. Because the mean values and the uncertainties of the interval parameters can be elected as the design variables, more information of the optimization results can be obtained by the present method than that obtained by the deterministic one. The present method is implemented for a truss structure and a frame structure. The numerical results show that the method is effective.

Keywords: Interval optimization; Interval parameter structure; Interval parameter; Interval extension of function; Deterministic optimization; Interval dynamic response

1. Introduction

The deterministic optimization [1,3–5,8] of structural behavior has been well developed for specified structural parameters and loading conditions. However, in most practical situations, the structural parameters and loads are uncertain, for example, there may be measurement inaccuracy or errors in the manufacturing process. Therefore, the concept of uncertainty plays an important role in the investigation of various engineering problems. The most common approach to problems of uncertainty is to model the structural parameters as random variables or fields. Under the circumstances, all information about the structural parameters is provided by the joint probability density function (or distribution function) of the structural parameters. Unfortunately, probabilistic model is not the only way one could describe the uncertainty, and uncertainty does not equal randomness. Indeed, probabilistic methods are not able to deliver reliable results at the required precision without sufficient experimental data to validate the assumptions made regarding the joint probability densities of the random variables or functions involved.

Since the mid-1960s, a new method called the interval analysis has appeared. Moore [10] and his co-workers, Alefeld and Herzberger [2] have done the pioneering work. The linear interval equations and nonlinear interval equations have been resolved. Hansen [9] in his book discussed the global optimization using interval analysis. Because of the complexity of the interval algorithm, it is difficult to deal with practical engineering problems. Recently, the interval analysis method has been used to deal with the static displacement, eigenvalue, and dynamic response analysis of the uncertain structures with interval parameters [6,7,11]. However, few papers can be found about the optimization of structures with interval parameters in engineering. Hence, it is necessary to develop an effective method to solve the optimal problems of structures with interval

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parameters. This paper presents an interval optimization method based on the interval analysis.

We will start with a brief review of the interval mathematics and the results of dynamic response analysis for structures with interval parameters [11], and then discuss the interval optimization model. Using the first-order Taylor expansion, matrix perturbation theory of dynamic response [12], and interval extension of functions, the interval optimization problem can be transformed into the approximate deterministic optimization one. The present method is implemented for a truss structure and a frame structure. Four numerical examples, the optimization of the dynamic response of a truss structure and a frame structure with interval parameters, are given. The numerical results are compared with those obtained by the deterministic optimization method.

2. Mathematical background

In structural analysis and design, some structural parameters have errors or uncertainties caused by manufacture, installation, computation or measurement. Therefore, it is very important to predict the errors resulted from the above-mentioned uncertainties in structural design. In interval mathematics, the errors or uncertainties are always denoted by intervals. Before we deal with the interval optimization problems, it is necessary to introduce some results in interval analysis [2,9,10].

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)^{\mathrm{T}}$ be a structural parameter vector with bound errors or uncertainties, where

$$\alpha_i \in \alpha_i^I = \left[\alpha_i^C - \Delta\alpha_i, \alpha_i^C + \Delta\alpha_i\right]$$

then

$$\alpha \in \alpha^I = \left[\alpha^C - \Delta\alpha, \alpha^C + \Delta\alpha\right]$$

where

$$\alpha^C = \left(\alpha_1^C, \alpha_2^C, \ldots, \alpha_m^C\right)^{\mathrm{T}}$$

and

$$\Delta\alpha = (\Delta\alpha_1, \Delta\alpha_2, \ldots, \Delta\alpha_m)^{\mathrm{T}}$$

Let $\underline{\alpha}$ and $\bar{\alpha}$ be the lower and upper bound vectors of the structural parameter vector α, respectively.

In interval mathematics, a subset of real numbers R of the form $[a_1, a_2] = \{t, a_1 \leqslant t \leqslant a_2 | a_1, a_2 \in R\}$ is called a closed real interval, denoted by $X^I = [\underline{X}, \overline{X}]$ where $\underline{X}$ and $\overline{X}$ are the lower and upper bounds, respectively. The set of all the closed real intervals is denoted by $I(R)$.

The mid-point and uncertainty of an interval X^I are defined as

$$X^C = m(X^I) = \left(\underline{X} + \overline{X}\right)/2 \tag{1}$$

and

$$\Delta X = \mathrm{rad}(X^I) = (\overline{X} - \underline{X})/2 \tag{2}$$

respectively.

A symmetric interval means an interval X^I in which $\underline{X} = -\overline{X}$.

Let $X^I = [\underline{X}, \overline{X}] \in I(R)$ be any interval, the relative uncertainty of X^I is defined as $\delta(X^I) = \frac{\Delta X}{|X^C|} = \frac{\overline{X} - \underline{X}}{|\overline{X} + \underline{X}|}$.

Let $X^I = [\underline{X}, \overline{X}]$ and $Y^I = [\underline{Y}, \overline{Y}] \in I(R)$ be any intervals, we say $X^I = Y^I$ if and only if $\underline{X} = \underline{Y}$ and $\overline{X} = \overline{Y}$. Let $X^I = [\underline{X}, \overline{X}]$ be any real interval, X^I is called point interval or degenerate interval if $\underline{X} = \overline{X}$, and then $X^I = [x, x] = x$.

We represent an n-dimensional interval vector as

$$\mathbf{X}^I = (X_1^I, X_2^I, \ldots, X_n^I)^{\mathrm{T}} \tag{3}$$

The set of all n-dimensional interval vectors is denoted by $I(R^n)$.

Similarly, the mid-vector and uncertainty of an interval vector can be defined as

$$\mathbf{X}^C = (X_1^C, X_2^C, \ldots, X_n^C)^{\mathrm{T}} \tag{4}$$

and

$$\Delta\mathbf{X} = (\Delta X_1, \Delta X_2, \ldots, \Delta X_n)^{\mathrm{T}} \tag{5}$$

where X_i^C and ΔX_i are given by Eqs. (1) and (2), respectively.

A matrix whose elements are intervals is called an interval matrix and denoted by $\mathbf{A}^I = [\underline{\mathbf{A}}, \overline{\mathbf{A}}]$, where $\underline{\mathbf{A}}$ is a matrix composed of the lower bounds of the intervals and $\overline{\mathbf{A}}$ is a matrix composed of the upper bounds of the intervals. The set of all interval matrices is denoted by $I(R^{m\times n})$. The mid-matrix and uncertainty of an interval matrix $\mathbf{A}^I$ are defined as

$$\mathbf{A}^C = \frac{\overline{\mathbf{A}} + \underline{\mathbf{A}}}{2} \quad \text{or} \quad a_{ij}^C = \frac{\overline{a_{ij}} + \underline{a_{ij}}}{2} \tag{6}$$

and

$$\Delta\mathbf{A} = \frac{\overline{\mathbf{A}} - \underline{\mathbf{A}}}{2} \quad \text{or} \quad \Delta a_{ij}^C = \frac{\overline{a_{ij}} - \underline{a_{ij}}}{2} \tag{7}$$

where $\mathbf{A}^C = (a_{ij}^c)$ and $\Delta\mathbf{A} = (\Delta a_{ij})$.

An arbitrary interval $X^I \in I(R)$ can be written as the following form

$$\begin{aligned} X^I &= X^C + \Delta X^I = X^C + \Delta X e_\Delta \\ &= [X^C - \Delta X, X^C + \Delta X] \end{aligned} \tag{8}$$

where $\Delta X^I = [-\Delta X, \Delta X]$ and $e_\Delta = [-1, 1]$.

Similar expressions exist for the interval vector and interval matrix. For $\mathbf{A}^I \in I(R^{m\times n})$, we have

$$\begin{aligned} \mathbf{A}^I &= \mathbf{A}^C + \Delta\mathbf{A}^I = \mathbf{A}^C + \Delta\mathbf{A} e_\Delta \\ &= [\mathbf{A}^C - \Delta\mathbf{A}, \mathbf{A}^C + \Delta\mathbf{A}] \end{aligned} \tag{9}$$

where $\Delta\mathbf{A}^I = [-\Delta\mathbf{A}, \Delta\mathbf{A}]$.

These basic quantities will play an important role in the following discussion.

Let $X^I = [\underline{x}, \bar{x}]$ and $Y^I = [\underline{y}, \bar{y}]$ be the interval numbers, respectively, then $X^I + Y^I$, $X^I - Y^I$, $X^I \times Y^I$ and X^I/Y^I are defined by the following formulas:

$$X^I + Y^I = [\underline{x}, \bar{x}] + [\underline{y}, \bar{y}] = [\underline{x} + \underline{y}, \bar{x} + \bar{y}] \tag{10}$$

$$X^I - Y^I = [\underline{x}, \bar{x}] - [\underline{y}, \bar{y}] = [\underline{x} - \bar{y}, \bar{x} - \underline{y}] \tag{11}$$

$$\begin{aligned} X^I \times Y^I &= [\underline{x}, \bar{x}] \times [\underline{y}, \bar{y}] \\ &= [\min(\underline{x}\underline{y}, \underline{x}\bar{y}, \bar{x}\underline{y}, \overline{xy}), \max(\underline{x}\underline{y}, \underline{x}\bar{y}, \bar{x}\underline{y}, \overline{xy})] \end{aligned} \tag{12}$$

$$\frac{X^I}{Y^I} = \frac{[\underline{x}, \bar{x}]}{[\underline{y}, \bar{y}]} = [\underline{x}, \bar{x}] \times \left[\frac{1}{\bar{y}}, \frac{1}{\underline{y}}\right], \quad 0 \not\subseteq Y^I \tag{13}$$

$$X^I \cap Y^I = [\max(\underline{x}, \underline{y}), \min(\bar{x}, \bar{y})] \tag{14}$$

$$X^I \cup Y^I = [\min(\underline{x}, \underline{y}), \max(\bar{x}, \bar{y})] \tag{15}$$

Let $\alpha \in R$ be any real number and $X^I = [\underline{x}, \bar{x}] = X^C + \Delta X e_\Delta \in I(R)$ be any real interval, then

$$\alpha X^I = X^I \alpha = X^C \alpha + \Delta X |\alpha| e_\Delta = \alpha X^C + |\alpha| \Delta X e_\Delta \tag{16}$$

Let $\mathbf{u} = (u_1, u_2, \ldots, u_n)^{\mathrm{T}} \in R^n$ be any real vector and $\mathbf{A}^I = [\underline{\mathbf{A}}, \overline{\mathbf{A}}] = \mathbf{A}^C + \Delta\mathbf{A} e_\Delta \in I(R^{n\times n})$ be any real interval matrix, then

$$\mathbf{A}^I \mathbf{u} = \mathbf{A}^C \mathbf{u} + \Delta\mathbf{A} |\mathbf{u}| e_\Delta \tag{17}$$

$$\mathbf{u}^{\mathrm{T}} \mathbf{A}^I = \mathbf{u}^{\mathrm{T}} \mathbf{A}^C + |\mathbf{u}|^{\mathrm{T}} \Delta\mathbf{A} e_\Delta \tag{18}$$

where $|\mathbf{u}| = (|u_1|, |u_2|, \ldots, |u_n|)^{\mathrm{T}}$, $e_\Delta = [-1, 1]$.

Let f be a real-valued function of n real variables $x_1, x_2, \ldots, x_n$. An interval extension of f means that an interval-valued function F of n interval variables $X_1^I, X_2^I, \ldots, X_n^I$, for all $x_i \in X_i^I$ $(i = 1, 2, \ldots, n)$, possesses the following property

$$F([x_1, x_1], [x_2, x_2], \ldots, [x_n, x_n]) = f(x_1, x_2, \ldots, x_n) \tag{19}$$

Given a rational function of real variables, we can replace the real variables by the corresponding interval variables and replace the real arithmetic operations by the corresponding interval arithmetic operations to obtain a rational interval function called a natural interval extension of the real rational function.

An interval function F is said to be inclusion monotonic if $X_i^I \subset Y_i^I (i = 1, 2, \ldots, n)$ implies

$$F(X_1^I, X_2^I, \ldots, X_n^I) \subset F(Y_1^I, Y_2^I, \ldots, Y_n^I) \tag{20}$$

It is obvious that interval arithmetic is inclusion monotonic. That is, if op denotes +,–,*,/ then $X_i^I \subset Y_i^I (i = 1, 2)$ implies

$$(X_1^I \mathrm{op} X_2^I) \subset (Y_1^I \mathrm{op} Y_2^I) \tag{21}$$

The interval extensions of a given function f are not unique. For example, two expressions for function g are given by

$$g^{(1)}(x, a) = \frac{ax}{1 - x}, \quad x \neq 1, \ a \neq 0 \tag{22}$$

$$g^{(2)}(x, a) = \frac{a}{\frac{1}{x} - 1}, \quad x \neq 1, \ a \neq 0 \tag{23}$$

Using $A^I = [0, 1]$ and $X^I = [2, 3]$ replace a and x, two possible evaluations can be obtained:

$$g^{(1)}([2, 3], [0, 1]) = \frac{[0, 1][2, 3]}{1 - [2, 3]} = [-3, 0]$$

$$g^{(2)}([2, 3], [0, 1]) = \frac{[0, 1]}{\frac{1}{[2,3]} - 1} = [-2, 0] \neq g^{(1)}([2, 3], [0, 1])$$

Both interval results contain the exact result of f for $x \in [2, 3]$ and $a \in [0, 1]$, which is $[-2, 0]$. The result for $g^{(2)}$ is precisely the range of g over the given sets, because X^I and A^I occur only once in the expression in $g^{(2)}$ [9]. It shows one important rule in interval calculation, that is, the least times the interval parameters appear, the sharper the interval is, which is important in interval calculations.

Irrational functions are treated as follows. Let f be a real irrational function of a real vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$. Assume that a rational approximation $r(\mathbf{x})$ is known such that $|f(\mathbf{x}) - r(\mathbf{x})| < \varepsilon$ for all $\mathbf{x}$ such that $a_i \leqslant x_i \leqslant b_i$ $(i = 1, \ldots, n)$ for some constants a_i and b_i. Then $f(X_1^I, X_2^I, \ldots, X_n^I) \subset r(X_1^I, X_2^I, \ldots, X_n^I) + [-\varepsilon, \varepsilon]$ for any intervals $X_i^I \subset [a_i, b_i]$ $(i = 1, \ldots, n)$. Thus the range of f over the region with $x_i \in X_i^I$ $(i = 1, \ldots, n)$ can be bounded by evaluating $r(X_1^I, \ldots, X_n^I)$ using interval arithmetic and adding the error bound $[-\varepsilon, \varepsilon]$.

This "interval evaluation" of the irrational function f is inclusion monotonic if the interval evaluation of r is inclusion monotonic. The result is an interval extension of f.

Then we have the general conclusion. Let $F(X_1^I, \ldots, X_n^I)$ be an inclusion monotonic interval extension of a real function $f(x_1, \ldots, x_n)$. Then $F(X_1^I, \ldots, X_n^I)$ contains the range of values of $f(x_1, \ldots, x_n)$ for all $x_i \in X_i^I$ $(i = 1, \ldots, n)$.

It is well known that typical structural design optimization problem resorts to finite element analysis in which the objective function or the constraint functions are not analytic. So it is difficult to get the exact interval solutions of the objective function or the constraint functions. We can resort to the first-order Taylor expansion to obtain the rational approximation of a complex function and then apply the natural interval extension to the rational approximation to get its interval solution. Thus the rational approximation of a complex function is a linear function of the variables and each variable appears only once, so the interval solution of the rational approximation we obtain is unique [9]. In order to justify the reasonability of this approach, we take a function we considered early as a simple example, that is, $g(x,a) = \frac{ax}{1-x}$, $x \neq 1$, $a \neq 0$. The exact solutions of the interval value for different interval variables are easy to calculated. Now we use Taylor expansion to expand the function about the mid-points of the interval variables to get the approximation of the interval value. In Table 1, we give the comparison for the interval value of the exact solution and the approximate solution for different interval variables, where δ is the relative uncertainty of a interval variable which is defined early in this paper. Suppose the mid-point and the uncertainty of the exact solution are denoted as f^C and Δf, respectively. Similarly, we denote the mid-point and the uncertainty of the approximate solution as g^C and Δg, respectively. The error of the mid-point is the value of $|(g^C - f^C)/f^c|$, and the error of the interval uncertainty is the value of $|(\Delta g - \Delta f)/\Delta f|$.

From Table 1, we can see that the errors of the mid-point and the interval uncertainty go up as the relative uncertainties of the interval variables increase. In fact, the relative uncertainties of the interval variables are small in practical engineering problems, so the approximate approach is acceptable for practical applications.

3. Interval characteristic matrices for structures with interval parameters

Assume that the interval parameters of the structures are denoted by

$$\mathbf{b} = (b_1, b_2, \ldots, b_m)^{\mathrm{T}} \in \mathbf{b}^I = (b_1^I, b_2^I, \ldots, b_m^I)^{\mathrm{T}} \tag{24}$$

The mean-value of the vector $\mathbf{b}^I$ is

$$\mathbf{b}^c = (b_1^c, b_2^c, \ldots, b_m^c)^{\mathrm{T}} \tag{25}$$

For each component of the vector, $b_j \in b_j^I = [\underline{b_j}, \overline{b_j}] = b_j^c + \Delta b_j e_j$, where $\Delta b_j = \frac{\overline{b_j} - \underline{b_j}}{2}$, $e_j = [-1, 1]$ and $j = 1, 2, \ldots, m$, m is the number of all parameters. The following discussions will be limited to the cases where the interval uncertainties of the interval parameters are small compared with the mean values, and the changes of parameters do not lead to the change of the shape of the element. For any $\mathbf{b} \in \mathbf{b}^I$, using the first-order Taylor expansion the characteristic matrices of the element can be expressed as

$$\begin{aligned}
\mathbf{K}_i^e(\mathbf{b}) &= \mathbf{K}_i^e(\mathbf{b}^c) + \sum_{j=1}^{m} \left(\frac{\partial \mathbf{K}_i^e(\mathbf{b})}{\partial b_j} \right)_{\mathbf{b}=\mathbf{b}^c} (b_j - b_j^c) \\
\mathbf{M}_i^e(\mathbf{b}) &= \mathbf{M}_i^e(\mathbf{b}^c) + \sum_{j=1}^{m} \left(\frac{\partial \mathbf{M}_i^e(\mathbf{b})}{\partial b_j} \right)_{\mathbf{b}=\mathbf{b}^c} (b_j - b_j^c) \\
\mathbf{C}_i^e(\mathbf{b}) &= \mathbf{C}_i^e(\mathbf{b}^c) + \sum_{j=1}^{m} \left(\frac{\partial \mathbf{C}_i^e(\mathbf{b})}{\partial b_j} \right)_{\mathbf{b}=\mathbf{b}^c} (b_j - b_j^c)
\end{aligned} \tag{26}$$

In general, it is difficult to express the stiffness, damping and mass matrices coefficients as explicit functions of design variables. To carry out the calculations of $\left(\frac{\partial \mathbf{K}_i^e(\mathbf{b})}{\partial b_j}\right)_{\mathbf{b}=\mathbf{b}^c}$, $\left(\frac{\partial \mathbf{M}_i^e(\mathbf{b})}{\partial b_j}\right)_{\mathbf{b}=\mathbf{b}^c}$ and $\left(\frac{\partial \mathbf{C}_i^e(\mathbf{b})}{\partial b_j}\right)_{\mathbf{b}=\mathbf{b}^c}$ by directly using the differential method is inconvenient. It is desirable to transform the differential approach into finite element perturbation. Let $\Delta \mathbf{K}_{ij}^e$, $\Delta \mathbf{C}_{ij}^e$ and $\Delta \mathbf{M}_{ij}^e$ be the increments of the stiffness, damping and mass matrices of the ith element resulting from the changes of the structural parameter ΔB_j, i.e.,

$$\begin{aligned}
\Delta \mathbf{K}_{ij}^e &= \mathbf{K}_i^e(b_1^c, \ldots, b_j^c + \Delta B_j, \ldots, b_m^c) \\
&\quad - \mathbf{K}_i^e(b_1^c, \ldots, b_j^c, \ldots, b_m^c) \\
\Delta \mathbf{M}_{ij}^e &= \mathbf{M}_i^e(b_1^c, \ldots, b_j^c + \Delta B_j, \ldots, b_m^c) \\
&\quad - \mathbf{M}_i^e(b_1^c, \ldots, b_j^c, \ldots, b_m^c) \\
\Delta \mathbf{C}_{ij}^e &= \mathbf{C}_i^e(b_1^c, \ldots, b_j^c + \Delta B_j, \ldots, b_m^c) \\
&\quad - \mathbf{C}_i^e(b_1^c, \ldots, b_j^c, \ldots, b_m^c)
\end{aligned} \tag{27}$$

Table 1
Comparison for the interval value of $g(x,a)$

Interval variables	δ	Exact solution	Approximate solution	Error of mid-point	Error of interval uncertainty
		f^I: [−1.03, −0.65]	g^I: [−1.02, −0.64]	0.71%	0.21%
$x^I = [2.4, 2.6]$	0.04	f^C: −0.8393	g^C: −0.8333		
$a^I = [0.4, 0.6]$	0.2	Δf: 0.1893	Δg: 0.1889		
		f^I: [−1.24, −0.48]	g^I: [−1.21, −0.46]	2.82%	0.84%
$x^I = [2.3, 2.7]$	0.08	f^C: −0.8575	g^C: −0.8333		
$a^I = [0.3, 0.7]$	0.4	Δf: 0.381	Δg: 0.3778		
		f^I: [−1.47, −0.31]	g^I: [−1.4, −0.27]	6.25%	1.92%
$x^I = [2.2, 2.8]$	0.12	f^C: −0.8889	g^C: −0.8333		
$a^I = [0.2, 0.8]$	0.6	Δf: 0.5778	Δg: 0.5667		

Then $\mathbf{K}^c_{i,j}$, $\mathbf{M}^c_{i,j}$ and $\mathbf{C}^c_{i,j}$, the approximation of $\left(\frac{\partial \mathbf{K}^e_i(\mathbf{b})}{\partial b_j}\right)_{\mathbf{b}=\mathbf{b}^c}$, $\left(\frac{\partial \mathbf{M}^e_i(\mathbf{b})}{\partial b_j}\right)_{\mathbf{b}=\mathbf{b}^c}$ and $\left(\frac{\partial \mathbf{C}^e_i(\mathbf{b})}{\partial b_j}\right)_{\mathbf{b}=\mathbf{b}^c}$ are as follows:

$$\mathbf{K}^c_{i,j} = \frac{\Delta \mathbf{K}^e_{ij}}{\Delta B_j} \quad \mathbf{M}^c_{i,j} = \frac{\Delta \mathbf{M}^e_{ij}}{\Delta B_j} \quad \mathbf{C}^c_{i,j} = \frac{\Delta \mathbf{C}^e_{ij}}{\Delta B_j} \tag{28}$$

Using the natural interval extension of function to Eq. (26), one can obtain the interval characteristic matrices

$$\begin{aligned}
\mathbf{K}^e_i(\mathbf{b}^I) &= \mathbf{K}^e_i(\mathbf{b}^c) + \sum_{j=1}^{m} \mathbf{K}^c_{i,j}(b^I_j - b^c_j) \\
\mathbf{M}^e_i(\mathbf{b}^I) &= \mathbf{M}^e_i(\mathbf{b}^c) + \sum_{j=1}^{m} \mathbf{M}^c_{i,j}(b^I_j - b^c_j) \\
\mathbf{C}^e_i(\mathbf{b}^I) &= \mathbf{C}^e_i(\mathbf{b}^c) + \sum_{j=1}^{m} \mathbf{C}^c_{i,j}(b^I_j - b^c_j)
\end{aligned} \tag{29}$$

The global stiffness and mass matrices are assembled by using the element matrices

$$\begin{aligned}
\mathbf{K}(\mathbf{b}) &= \sum_{i=1}^{n} \mathbf{K}^e_i(\mathbf{b}) = \mathbf{K}(\mathbf{b}^c) + \Delta\mathbf{K}(\mathbf{b}) \\
\mathbf{M}(\mathbf{b}) &= \sum_{i=1}^{n} \mathbf{M}^e_i(\mathbf{b}) = \mathbf{M}(\mathbf{b}^c) + \Delta\mathbf{M}(\mathbf{b}) \\
\mathbf{C}(\mathbf{b}) &= \sum_{i=1}^{n} \mathbf{C}^e_i(\mathbf{b}) = \mathbf{C}(\mathbf{b}^c) + \Delta\mathbf{C}(\mathbf{b})
\end{aligned} \tag{30}$$

where

$$\begin{aligned}
\mathbf{K}(\mathbf{b}^c) &= \sum_{i=1}^{n} \mathbf{K}^e_i(\mathbf{b}^c) \quad \Delta\mathbf{K}(\mathbf{b}) = \sum_{i=1}^{n}\sum_{j=1}^{m} \mathbf{K}^c_{i,j}\Delta b_j \\
\mathbf{M}(\mathbf{b}^c) &= \sum_{i=1}^{n} \mathbf{M}^e_i(\mathbf{b}^c) \quad \Delta\mathbf{M}(\mathbf{b}) = \sum_{i=1}^{n}\sum_{j=1}^{m} \mathbf{M}^c_{i,j}\Delta b_j \\
\mathbf{C}(\mathbf{b}^c) &= \sum_{i=1}^{n} \mathbf{C}^e_i(\mathbf{b}^c) \quad \Delta\mathbf{C}(\mathbf{b}) = \sum_{i=1}^{n}\sum_{j=1}^{m} \mathbf{C}^c_{i,j}\Delta b_j
\end{aligned} \tag{31}$$

where n is the total number of the elements. It should be pointed out that in Eq. (30), the element characteristic matrices should be expanded by FEM rules before forming the global matrices. Applying the natural interval extension of function to Eq. (30), one can obtain the interval matrices as follows:

$$\begin{aligned}
\mathbf{K}(\mathbf{b}^I) &= \mathbf{K}(\mathbf{b}^c) + \Delta\mathbf{K}(\mathbf{b}^I) \\
\mathbf{M}(\mathbf{b}^I) &= \mathbf{M}(\mathbf{b}^c) + \Delta\mathbf{M}(\mathbf{b}^I) \\
\mathbf{C}(\mathbf{b}^I) &= \mathbf{C}(\mathbf{b}^c) + \Delta\mathbf{C}(\mathbf{b}^I)
\end{aligned} \tag{32}$$

where

$$\begin{aligned}
\Delta\mathbf{K}(\mathbf{b}^I) &= \sum_{i=1}^{n}\sum_{j=1}^{m} \mathbf{K}^c_{i,j}(b^I_j - b^c_j) \\
\Delta\mathbf{M}(\mathbf{b}^I) &= \sum_{i=1}^{n}\sum_{j=1}^{m} \mathbf{M}^c_{i,j}(b^I_j - b^c_j) \\
\Delta\mathbf{C}(\mathbf{b}^I) &= \sum_{i=1}^{n}\sum_{j=1}^{m} \mathbf{C}^c_{i,j}(b^I_j - b^c_j)
\end{aligned} \tag{33}$$

The damping coefficients are taken as the Rayleigh damping, i.e., $\mathbf{C} = \alpha\mathbf{K} + \beta\mathbf{M}$, where α and β are the coefficients, which can be taken as the structural parameters.

4. Dynamic response analysis of structures with interval parameters

4.1. Perturbation analysis of the dynamic response of deterministic system

The vibration equation of n-degrees-of-freedom systems can be given as follows:

$$\mathbf{M}(\mathbf{b})\ddot{\mathbf{x}} + \mathbf{C}(\mathbf{b})\dot{\mathbf{x}} + \mathbf{K}(\mathbf{b})\mathbf{x} = \mathbf{P}(\mathbf{t}) \tag{34}$$

If the design variables have some perturbations $\varepsilon\Delta\mathbf{b}$, i.e., $\mathbf{b} = \mathbf{b}^c + \varepsilon\Delta\mathbf{b} \in \mathbf{b}^I$, then the characteristic matrices can be written as [12],

$$\begin{aligned}
\mathbf{M}(\mathbf{b}) &= \mathbf{M}(\mathbf{b}^c) + \varepsilon\mathbf{M}_1 \\
\mathbf{C}(\mathbf{b}) &= \mathbf{C}(\mathbf{b}^c) + \varepsilon\mathbf{C}_1 \\
\mathbf{K}(\mathbf{b}) &= \mathbf{K}(\mathbf{b}^c) + \varepsilon\mathbf{K}_1
\end{aligned} \tag{35}$$

and the responses are

$$\begin{aligned}
\mathbf{x} &= \mathbf{x}_0 + \varepsilon\mathbf{x}_1 + \varepsilon^2\mathbf{x}_2 + \cdots \\
\dot{\mathbf{x}} &= \dot{\mathbf{x}}_0 + \varepsilon\dot{\mathbf{x}}_1 + \varepsilon^2\dot{\mathbf{x}}_2 + \cdots \\
\ddot{\mathbf{x}} &= \ddot{\mathbf{x}}_0 + \varepsilon\ddot{\mathbf{x}}_1 + \varepsilon^2\ddot{\mathbf{x}}_2 + \cdots
\end{aligned} \tag{36}$$

Then Eq. (34) becomes

$$\begin{aligned}
&(\mathbf{M}(\mathbf{b}^c) + \varepsilon\mathbf{M}_1)(\ddot{\mathbf{x}}_0 + \varepsilon\ddot{\mathbf{x}}_1 + \cdots) + (\mathbf{C}(\mathbf{b}^c) + \varepsilon\mathbf{C}_1) \\
&\quad \times (\dot{\mathbf{x}}_0 + \varepsilon\dot{\mathbf{x}}_1 + \cdots) + (\mathbf{K}(\mathbf{b}^c) + \varepsilon\mathbf{K}_1)(\mathbf{x}_0 + \varepsilon\mathbf{x}_1 + \cdots) \\
&\quad = \mathbf{P}(t)
\end{aligned} \tag{37}$$

Expanding and equating the coefficients with the same power in Eq. (37), one can obtain

$$\varepsilon^0 : \mathbf{M}(\mathbf{b}^c)\ddot{\mathbf{x}}_0 + \mathbf{C}(\mathbf{b}^c)\dot{\mathbf{x}}_0 + \mathbf{K}(\mathbf{b}^c)\mathbf{x}_0 = \mathbf{P}(t) \tag{38}$$

$$\begin{aligned}
\varepsilon^1 : &\mathbf{M}(\mathbf{b}^c)\ddot{\mathbf{x}}_1 + \mathbf{C}(\mathbf{b}^c)\dot{\mathbf{x}}_1 + \mathbf{K}(\mathbf{b}^c)\mathbf{x}_1 \\
&= -(\mathbf{M}_1\ddot{\mathbf{x}}_0 + \mathbf{C}_1\dot{\mathbf{x}}_0 + \mathbf{K}_1\mathbf{x}_0)
\end{aligned} \tag{39}$$

From Eq. (38), one can obtain the dynamic response of the original system. However, it is difficult to acquire the perturbation solutions from Eq. (39). If the changes of the structural parameters are small, one can expand $\mathbf{M}_1$,

$\mathbf{C}_1$, $\mathbf{K}_1$, $\mathbf{x}_1$, $\dot{\mathbf{x}}_1$ and $\ddot{\mathbf{x}}_1$ around the mean value of the parameters, that is,

$$\mathbf{x}_1 = \sum_{j=1}^{m} \mathbf{x}_{0,j}\Delta b_j \quad \dot{\mathbf{x}}_1 = \sum_{j=1}^{m} \dot{\mathbf{x}}_{0,j}\Delta b_j \quad \ddot{\mathbf{x}}_1 = \sum_{j=1}^{m} \ddot{\mathbf{x}}_{0,j}\Delta b_j$$
$$\mathbf{M}_1 = \sum_{j=1}^{m} \mathbf{M}_{0,j}(b_j - b_j^c) \quad \mathbf{C}_1 = \sum_{j=1}^{m} \mathbf{C}_{0,j}(b_j - b_j^c)$$
$$\mathbf{K}_1 = \sum_{j=1}^{m} \mathbf{K}_{0,j}(b_j - b_j^c) \tag{40}$$

in which

$$\mathbf{x}_{0,j} = \frac{\partial \mathbf{x}_0}{\partial b_j} \quad \dot{\mathbf{x}}_{0,j} = \frac{\partial \dot{\mathbf{x}}_0}{\partial b_j} \quad \ddot{\mathbf{x}}_{0,j} = \frac{\partial \ddot{\mathbf{x}}_0}{\partial b_j}$$

$$\mathbf{M}_{0,j} = \frac{\partial \mathbf{M}(\mathbf{b}^c)}{\partial b_j} \quad \mathbf{C}_{0,j} = \frac{\partial \mathbf{C}(\mathbf{b}^c)}{\partial b_j} \quad \mathbf{K}_{0,j} = \frac{\partial \mathbf{K}(\mathbf{b}^c)}{\partial b_j}$$

Substituting Eq. (40) into Eq. (39), one can obtain

$$\sum_{j=1}^{m}(\mathbf{M}(\mathbf{b}^c)\ddot{\mathbf{x}}_{0,j} + \mathbf{C}(\mathbf{b}^c)\dot{\mathbf{x}}_{0,j} + \mathbf{K}(\mathbf{b}^c)\mathbf{x}_{0,j})\Delta b_j = -\sum_{j=1}^{m}(\mathbf{M}_{0,j}\ddot{\mathbf{x}}_0 + \mathbf{C}_{0,j}\dot{\mathbf{x}}_0 + \mathbf{K}_{0,j}\mathbf{x}_0)\Delta b_j \tag{41}$$

From Eq. (41), one can get

$$\mathbf{M}(\mathbf{b}^c)\ddot{\mathbf{x}}_{0,j} + \mathbf{C}(\mathbf{b}^c)\dot{\mathbf{x}}_{0,j} + \mathbf{K}(\mathbf{b}^c)\mathbf{x}_{0,j} = -(\mathbf{M}_{0,j}\ddot{\mathbf{x}}_0 + \mathbf{C}_{0,j}\dot{\mathbf{x}}_0 + \mathbf{K}_{0,j}\mathbf{x}_0) \tag{42}$$

It is easy to get the solutions of those equations by the normal numerical integral methods as Wilson-θ or Newmark etc. Substituting the solutions into Eq. (36), the response perturbation part is obtained, and then the response solution of the perturbed system is

$$\mathbf{x} = \mathbf{x}_0 + \varepsilon\mathbf{x}_1 \tag{43}$$

4.2. Interval dynamic response of structures with interval parameters

Using the interval extension of function to Eq. (34), one has

$$\mathbf{M}(\mathbf{b}^I)\ddot{\mathbf{x}} + \mathbf{C}(\mathbf{b}^I)\dot{\mathbf{x}} + \mathbf{K}(\mathbf{b}^I)\mathbf{x} = \mathbf{P}(t) \tag{44}$$

where

$$\mathbf{M}(\mathbf{b}^I) = \{\mathbf{M}(\mathbf{b})|\underline{\mathbf{b}} \leqslant \mathbf{b} \leqslant \bar{\mathbf{b}}\}$$
$$\mathbf{C}(\mathbf{b}^\mathbf{I}) = \{\mathbf{C}(\mathbf{b})|\underline{\mathbf{b}} \leqslant \mathbf{b} \leqslant \bar{\mathbf{b}}\} \tag{45}$$
$$\mathbf{K}(\mathbf{b}^\mathbf{I}) = \{\mathbf{K}(\mathbf{b})|\underline{\mathbf{b}} \leqslant \mathbf{b} \leqslant \bar{\mathbf{b}}\}$$

It is the basic problem for given interval characteristic matrices, $\mathbf{M}(\mathbf{b}^I)$, $\mathbf{C}(\mathbf{b}^I)$, $\mathbf{K}(\mathbf{b}^I)$ and $\mathbf{P}(t)$, to find all possible $\mathbf{x}$ satisfying Eq. (34), that is, to obtain $\mathbf{x} \in \mathbf{x}^I = [\underline{\mathbf{x}}, \bar{\mathbf{x}}]$ where

$$\underline{\mathbf{x}} = \min\{\mathbf{x}|\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{P}(t), \mathbf{M} \in \mathbf{M}(\mathbf{b}^I), \mathbf{C} \in \mathbf{C}(\mathbf{b}^I), \mathbf{K} \in \mathbf{K}(\mathbf{b}^I)\}$$

$$\bar{\mathbf{x}} = \max\{\mathbf{x}|\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{P}(t), \mathbf{M} \in \mathbf{M}(\mathbf{b}^I), \mathbf{C} \in \mathbf{C}(\mathbf{b}^I), \mathbf{K} \in \mathbf{K}(\mathbf{b}^I)\}$$

Using Eq. (32), Eq. (44) becomes

$$(\mathbf{M}(\mathbf{b}^c) + \Delta\mathbf{M}(\mathbf{b}^I))\ddot{\mathbf{x}} + (\mathbf{C}(\mathbf{b}^c) + \Delta\mathbf{C}(\mathbf{b}^I))\dot{\mathbf{x}} + (\mathbf{K}(\mathbf{b}^c) + \Delta\mathbf{K}(\mathbf{b}^I))\mathbf{x} = \mathbf{P}(t) \tag{46}$$

For any $\mathbf{b} = \mathbf{b}^c + \delta\mathbf{b} \in \mathbf{b}^I$, there is a group of $\delta\mathbf{M}$, $\delta\mathbf{C}$, and $\delta\mathbf{K}$, which satisfy

$$\underline{\Delta\mathbf{M}} \leqslant \delta\mathbf{M} \leqslant \overline{\Delta\mathbf{M}} \quad \underline{\Delta\mathbf{C}} \leqslant \delta\mathbf{C} \leqslant \overline{\Delta\mathbf{C}} \quad \underline{\Delta\mathbf{K}} \leqslant \delta\mathbf{K} \leqslant \overline{\Delta\mathbf{K}} \tag{47}$$

and the vibration equation is

$$(\mathbf{M}(\mathbf{b}^c) + \delta\mathbf{M})\ddot{\mathbf{x}} + (\mathbf{C}(\mathbf{b}^c) + \delta\mathbf{C})\dot{\mathbf{x}} + (\mathbf{K}(\mathbf{b}^c) + \delta\mathbf{K})\mathbf{x} = \mathbf{P}(t) \tag{48}$$

By neglecting the higher order terms, from Eq. (43), one can obtain

$$\mathbf{x} = \mathbf{x}_0 + \delta\mathbf{x}$$
$$\delta\mathbf{x} = \sum_{j=1}^{m} \mathbf{x}_{0,j}(b_j - b_j^c) \tag{49}$$

in which $\mathbf{x}_0$ and $\mathbf{x}_{0,j}$ are obtained by solving Eq. (38) and Eq. (42). Eq. (46) are equivalent to Eq. (48) with the constraints Eq. (47). Therefore, using the interval extension of function to Eq. (49), one gets

$$\mathbf{x}^I = \mathbf{x}_0 + \Delta\mathbf{x}^I$$
$$\Delta\mathbf{x}^I = \sum_{j=1}^{m} \mathbf{x}_{0,j}(b_j^I - b_j^c) = \sum_{j=1}^{m} \mathbf{x}_{0,j}\Delta b_j e_j = \sum_{j=1}^{m} |\mathbf{x}_{0,j}\Delta b_j|[-1, 1] = \Delta\mathbf{x}[-1, 1] \tag{50}$$

where $\Delta\mathbf{x} = \sum_{j=1}^{m} |\mathbf{x}_{0,j}\Delta b_j|$, and the upper and lower bounds of the dynamic response will be

$$\bar{\mathbf{x}} = \mathbf{x}_0 + \Delta\mathbf{x}$$
$$\underline{\mathbf{x}} = \mathbf{x}_0 - \Delta\mathbf{x} \tag{51}$$

From Eq. (51), one can obtain the interval responses which are symmetrical about the mean value $\mathbf{x}_0$, and the intervals of dynamic responses will be sharper if the interval operations are used at the least times to calculate $\mathbf{x}_{0,j}$ in Eq. (42).

5. Interval optimization model

If the structural parameters are assumed to be interval variables, the objective function and the constraint conditions of the optimization problems are interval. Therefore, an interval optimization problem can be described as follows:

$$\begin{aligned} &\min \quad f(\mathbf{X}^I) = f(X_1^I, X_2^I, \ldots, X_n^I) \\ &\text{s.t.} \quad \begin{cases} p_i(\mathbf{X}^I) \leqslant 0 & (i = 1, 2, \ldots, m) \\ q_j(\mathbf{X}^I) = 0 & (j = 1, 2, \ldots, l) \end{cases} \end{aligned} \tag{52}$$

where $\mathbf{X}^I = (X_1^I, X_2^I, \ldots, X_n^I)^{\mathrm{T}}$ is the interval parameter vector of the structure, $f(\mathbf{X}^I)$ is the interval objective function, and $p_i(\mathbf{X}^I)$ and $q_j(\mathbf{X}^I)$ are the interval constraint conditions.

It should be noted that it is difficult to solve the interval optimization problem described in Eq. (52) directly [9]. In order to simplify the interval optimization problems, we transform it into approximate deterministic one. To this end, using the Taylor expansion to expand $f(\mathbf{X}^I)$ about the mid-vector of the interval vector $\mathbf{X}^I$ and neglecting the higher order terms, and considering Eq. (16), one has

$$\begin{aligned} f(\mathbf{X}^I) &= f(\mathbf{X}^c) + \sum_{i=1}^{n} \frac{\partial f(\mathbf{X}^c)}{\partial x_i}(X_i^I - X_i^c) \\ &= f(\mathbf{X}^c) + \sum_{i=1}^{n} \frac{\partial f(\mathbf{X}^c)}{\partial x_i}\Delta X_i e_\Delta \\ &= f(\mathbf{X}^c) + \sum_{i=1}^{n} \left|\frac{\partial f(\mathbf{X}^c)}{\partial x_i}\Delta X_i\right| e_\Delta \\ &= f(\mathbf{X}^c) + \left(\sum_{i=1}^{n} \left|\frac{\partial f(\mathbf{X}^c)}{\partial x_i}\Delta X_i\right|\right) e_\Delta \\ &= [f(\mathbf{X}^c), f(\mathbf{X}^c)] \\ &\quad + \left[-\sum_{i=1}^{n}\left|\frac{\partial f(\mathbf{X}^c)}{\partial x_i}\Delta X_i\right|, \sum_{i=1}^{n}\left|\frac{\partial f(\mathbf{X}^c)}{\partial x_i}\Delta X_i\right|\right] \\ &= \left[f(\mathbf{X}^c) - \sum_{i=1}^{n}\left|\frac{\partial f(\mathbf{X}^c)}{\partial x_i}\Delta X_i\right|, f(\mathbf{X}^c)\right. \\ &\quad \left. + \sum_{i=1}^{n}\left|\frac{\partial f(\mathbf{X}^c)}{\partial x_i}\Delta X_i\right|\right] \end{aligned} \tag{53}$$

Similarly, the constraint conditions can be obtained

$$\begin{aligned} p_i(\mathbf{X}^I) &= p_i(\mathbf{X}^c) + \sum_{k=1}^{n} \frac{\partial p_i(\mathbf{X}^c)}{\partial x_k}(X_k^I - X_k^c) \\ &= p_i(\mathbf{X}^c) + \sum_{k=1}^{n} \frac{\partial p_i(\mathbf{X}^c)}{\partial x_k}\Delta X_k e_\Delta \\ &= p_i(\mathbf{X}^c) + \sum_{k=1}^{n} \left|\frac{\partial p_i(\mathbf{X}^c)}{\partial x_k}\Delta X_k\right| e_\Delta \\ &= p_i(\mathbf{X}^c) + \left(\sum_{k=1}^{n} \left|\frac{\partial p_i(\mathbf{X}^c)}{\partial x_k}\Delta X_k\right|\right) e_\Delta \\ &= [p_i(\mathbf{X}^c), p_i(\mathbf{X}^c)] \\ &\quad + \left[-\sum_{k=1}^{n}\left|\frac{\partial p_i(\mathbf{X}^c)}{\partial x_k}\Delta X_k\right|, \sum_{k=1}^{n}\left|\frac{\partial p_i(\mathbf{X}^c)}{\partial x_k}\Delta X_k\right|\right] \\ &= \left[p_i(\mathbf{X}^c) - \sum_{k=1}^{n}\left|\frac{\partial p_i(\mathbf{X}^c)}{\partial x_k}\Delta X_k\right|, p_i(\mathbf{X}^c)\right. \\ &\quad \left. + \sum_{k=1}^{n}\left|\frac{\partial p_i(\mathbf{X}^c)}{\partial x_k}\Delta X_k\right|\right] \quad (i = 1, 2, \ldots, m) \end{aligned} \tag{54}$$

$$\begin{aligned} q_j(\mathbf{X}^I) &= q_j(\mathbf{X}^c) + \sum_{k=1}^{n} \frac{\partial q_j(\mathbf{X}^c)}{\partial x_k}(X_k^I - X_k^c) \\ &= q_j(\mathbf{X}^c) + \sum_{k=1}^{n} \frac{\partial q_j(\mathbf{X}^c)}{\partial x_k}\Delta X_k e_\Delta \\ &= q_j(\mathbf{X}^c) + \sum_{k=1}^{n} \left|\frac{\partial q_j(\mathbf{X}^c)}{\partial x_k}\Delta X_k\right| e_\Delta \\ &= q_j(\mathbf{X}^c) + \left(\sum_{k=1}^{n} \left|\frac{\partial q_j(\mathbf{X}^c)}{\partial x_k}\Delta X_k\right|\right) e_\Delta \\ &= [q_j(\mathbf{X}^c), q_j(\mathbf{X}^c)] \\ &\quad + \left[-\sum_{k=1}^{n}\left|\frac{\partial q_j(\mathbf{X}^c)}{\partial x_k}\Delta X_k\right|, \sum_{k=1}^{n}\left|\frac{\partial q_j(\mathbf{X}^c)}{\partial x_k}\Delta X_k\right|\right] \\ &= \left[q_j(\mathbf{X}^c) - \sum_{k=1}^{n}\left|\frac{\partial q_j(\mathbf{X}^c)}{\partial x_k}\Delta X_k\right|, q_j(\mathbf{X}^c)\right. \\ &\quad \left. + \sum_{k=1}^{n}\left|\frac{\partial q_j(\mathbf{X}^c)}{\partial x_k}\Delta X_k\right|\right] \quad (j = 1, 2, \ldots, l) \end{aligned} \tag{55}$$

where $\mathbf{X}^I = (X_1^I, X_2^I, \ldots, X_n^I)^{\mathrm{T}}$, $\mathbf{X}^c = (X_1^c, X_2^c, \ldots, X_n^c)^{\mathrm{T}}$, and $e_\Delta = [-1, 1]$.

Then the interval optimization problem can be transformed into the approximate deterministic optimization one as follows:

$$\begin{aligned} &\min \quad \left[f(\mathbf{X}^c) + \sum_{i=1}^{n}\left|\frac{\partial f(\mathbf{X}^c)}{\partial x_i}\right|\Delta X_i\right] \\ &\text{s.t.} \quad \begin{cases} p_i(\mathbf{X}^c) + \sum_{k=1}^{n}\left|\frac{\partial p_i(\mathbf{X}^c)}{\partial x_k}\right|\Delta X_k \leqslant 0 & (i = 1, 2, \ldots, m) \\ q_j(\mathbf{X}^c) = 0 & (j = 1, 2, \ldots, l) \\ \sum_{k=1}^{n}\left|\frac{\partial q_j(\mathbf{X}^c)}{\partial x_k}\right|\Delta X_k = 0 & (j = 1, 2, \ldots, l) \\ -\Delta X_k \leqslant 0 & (k = 1, 2, \ldots, n) \end{cases} \end{aligned} \tag{56}$$

6. Application in the truss structure

In this section, we will apply the interval optimization method presented by Section 5 to the truss structure

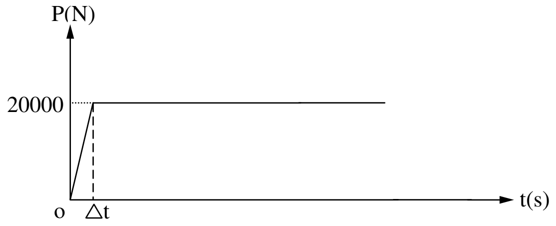

Fig. 1. The step excitation.

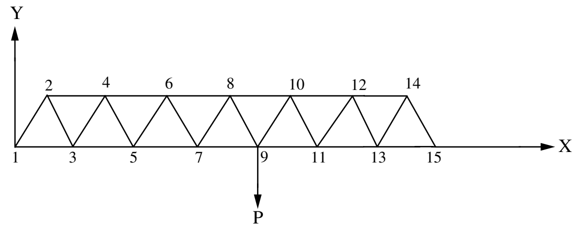

Fig. 2. A truss structure.

with 30 degrees of freedom as shown in Fig. 2. Suppose the step excitation as shown in Fig. 1 is applied to node 9 along the y-negative direction. The goal of the interval optimization is to determine the mid-points and uncertainties of the design variables $\{E, A, \rho\}$, so that the displacement amplitude at the y-direction of node 9 takes minimum and has a relative narrow interval. We can rewrite all the design variables as a three-dimensional vector

$$\mathbf{X} = \{X_1, X_2, X_3\}^{\mathrm{T}} = \{E, A, \rho\}^{\mathrm{T}} \tag{57}$$

where E is the Young's module, A the crossing section area, ρ the mass density.

The deterministic optimization problem is given by

$$\begin{aligned} &\min \quad \Psi(\mathbf{X}) = \mathbf{x}(\mathbf{X}) \\ &\text{s.t.} \quad \begin{cases} g_i(\mathbf{X}) = x_i^L - X_i \leqslant 0 & (i = 1, 2, 3) \\ g_{i+3}(\mathbf{X}) = X_i - x_i^U \leqslant 0 & (i = 1, 2, 3) \end{cases} \end{aligned} \tag{58}$$

where x_i^L and x_i^U are the ranges of the design variable x_i, respectively.

The interval optimization problem for the truss structure is

$$\begin{aligned} &\min \ \Psi(\mathbf{X}^I) = \mathbf{x}(\mathbf{X}^I) \\ &\text{s.t.} \quad g_i(\mathbf{X}^I) \leqslant 0 \quad (i = 1, 2, \ldots, 6) \end{aligned} \tag{59}$$

The approximate deterministic optimization problem is

$$\begin{aligned} &\min \quad \bar{\mathbf{x}} = \left[\mathbf{x}_0(\mathbf{X}^c) + \sum_{j=1}^{3} \left| \frac{\partial \mathbf{x}_0(\mathbf{X}^c)}{\partial x_j} \Delta X_j \right| \right] \\ &\text{s.t.} \quad \begin{cases} x_i^L - X_i^c + \Delta X_i \leqslant 0 & (i = 1, 2, 3) \\ X_i^c - x_i^U + \Delta X_i \leqslant 0 & (i = 1, 2, 3) \\ -\Delta X_k \leqslant 0 & (k = 1, 2, 3) \end{cases} \end{aligned} \tag{60}$$

Example 1. Consider a truss structure hinged at both ends shown in Fig. 2. The displacement amplitude at the y-direction of the node 9 is 5.347E−4 m, which can be calculated with the mean value of the parameters (for example, $E = 2.1\mathrm{E}11$ N/m^2, $A = 16\mathrm{E}{-4}$ m^2, and $\rho = 7800.0$ kg/m^3). In order to reduce the displacement amplitude, interval optimization method can be used. In this example we suppose the uncertainties of the design variables are specified in advance, that is, $\Delta E = 2.0\mathrm{E}10$ N/m^2, $\Delta A = 4.0\mathrm{E}{-5}$ m^2, and $\Delta\rho = 780.0$ kg/m^3.

There are three optimization parameters: E^c, A^c, ρ^c.

The interval optimization is to optimize the displacement amplitude at the y-direction of the node 9. The ranges of the optimization parameters E^c, A^c and ρ^c are 1.5E11–2.5E11 N/m^2, 4E−4–25E−4 m^2, and 6000.0–8000.0 kg/m^3. Using Lagrange optimal algorithm, the approximate deterministic optimization can be solved. The results of the interval optimization are shown in Table 2. For comparison, the results of the deterministic optimization are also listed in Table 2.

Table 2
Comparison of results of deterministic optimization and interval optimization with three optimization parameters

X_i^c	Initial values	Deterministic optimization values	Interval optimization
			Mid-point values
E^c	1.5E11	2.5E11	2.5E11
A^c	4.7E−4	2.496E−3	2.499E−3
ρ^c	7000.0	7900.0	7204.35
ΔX_i(*specified*)			Uncertainties
ΔE	2.0E10	0.0	2.0E10
ΔA	4.0E−5	0.0	4.0E−5
$\Delta\rho$	780.0	0.0	780.0
min $\Psi(\mathbf{X})$		2.879161E−4	[1.3679E−4, 4.3827E−4]

Table 3
Comparison of results of deterministic optimization and interval optimization with six optimization parameters

X_i^c	Initial values	Deterministic optimization values	Interval optimization
			Mid-point values
E^c	1.5E11	2.5E11	2.35141E11
A^c	4.8E−4	2.496E−3	2.5E−3
ρ^c	7800.0	7900.0	7978.77
ΔX_i			Uncertainties
ΔE	2.2E10	0.0	1.70099E10
ΔA	2.4E−4	0.0	4.00016E−5
$\Delta\rho$	780.0	0.0	636.24
$\min \Psi(\mathbf{X})$		2.879161E−4	[2.1125E−4, 3.9998E−4]

From Table 2, it can be seen that the displacement at the y-direction of the node 9 with the deterministic optimization is 2.879161E−4 m, while the corresponding result with the interval optimization is an interval value [1.3679E−4 m, 4.3827E−4 m].

Example 2. Consider the same truss structure shown in Fig. 2. In this example, we suppose the uncertainties of the design variables are also the optimization parameters. Thus, there are six optimization parameters: E^c, A^c, ρ^c, ΔE, ΔA, $\Delta\rho$.

The interval optimization is also to optimize the displacement amplitude at the y-direction of the node 9. The ranges of the optimization parameters E^c, A^c, ρ^c, ΔE, ΔA, and $\Delta\rho$ are 1.5E11–2.5E11 N/m^2, 4E−4–25E−4 m^2, 6000.0–8000.0 kg/m^3, 1.5E10–2.5E10 N/m^2, 4E−5–25E−5 m^2, and 600.0–800.0 kg/m^3.

Using the Lagrange optimal algorithm, the approximate deterministic optimization can be solved. The results of the interval optimization are shown in Table 3. For comparison, the results of the deterministic optimization are also listed in Table 3.

From Table 3, it can be seen that the displacement amplitude of the deterministic optimization is 2.879161E−4 m, while the result of the interval optimization is an interval value [2.112528E−4 m, 3.999878E−4 m].

From the Tables 2 and 3 it can be seen that, because we take more variables as optimization parameters, the interval value of the objective function obtained by Example 2 is sharper than that obtained by the Example 1.

7. Application in the frame structure

In this section, we will apply the interval optimization method presented by Section 5 to the frame structure with 30 degrees of freedom as shown in Fig. 3. Suppose the sine excitation with frequency ($\omega = 90$ s^{-1}) at node 10 is along the x-positive direction, and the amplitude of the load is 3000 N. The goal of the interval optimization is to determine the mid-points and uncertainties of the design variables $\{B, H\}$, so that the displacement amplitude at the x-direction of node 10 takes minimum and has a relative narrow interval. We can rewrite all the design variables as a two-dimensional vector

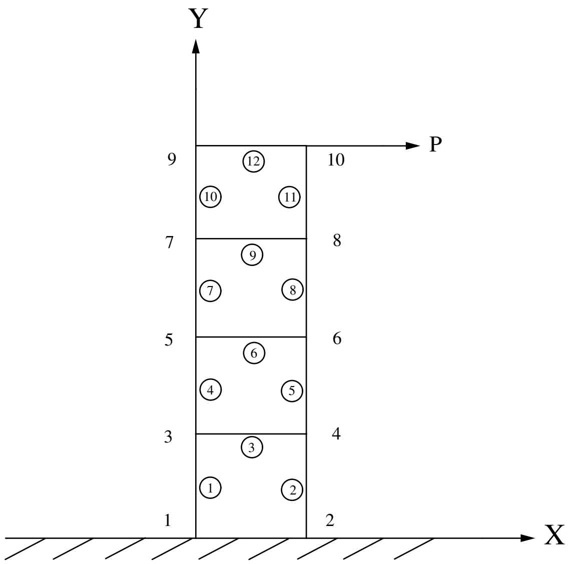

Fig. 3. A frame structure.

$$\mathbf{X} = \{X_1, X_2\}^{\mathrm{T}} = \{B, H\}^{\mathrm{T}} \tag{61}$$

where B is the width of the cross section of the beam element, H the height.

The deterministic optimization problem is given by

$$\begin{aligned} &\min \quad \Psi(\mathbf{X}) = \mathbf{x}(\mathbf{X}) \\ &\text{s.t.} \quad \begin{cases} g_i(\mathbf{X}) = x_i^L - X_i \leqslant 0 & (i = 1, 2) \\ g_{i+2}(\mathbf{X}) = X_i - x_i^U \leqslant 0 & (i = 1, 2) \end{cases} \end{aligned} \tag{62}$$

where x_i^L and x_i^U are the ranges of the design variable x_i, respectively.

The interval optimization problem for the frame structure is

$$\begin{aligned} &\min \ \Psi(\mathbf{X}^I) = \mathbf{x}(\mathbf{X}^I) \\ &\text{s.t.} \quad g_i(\mathbf{X}^I) \leqslant 0 \quad (i = 1, \ldots, 4) \end{aligned} \tag{63}$$

The approximate deterministic optimization problem is

$$\begin{aligned} &\min \ \bar{\mathbf{x}} = \left[\mathbf{x}_0(\mathbf{X}^c) + \sum_{j=1}^{2} \left| \frac{\partial \mathbf{x}_0(\mathbf{X}^c)}{\partial x_j} \Delta X_j \right| \right] \\ &\text{s.t.} \ \begin{cases} x_i^L - X_i^c + \Delta X_i \leqslant 0 & (i = 1, 2) \\ X_i^c - x_i^U + \Delta X_i \leqslant 0 & (i = 1, 2) \\ -\Delta X_k \leqslant 0 & (k = 1, 2) \end{cases} \end{aligned} \tag{64}$$

Example 3. Consider the frame structure shown in Fig. 3. The displacement amplitude at the x-direction of the node 10 is 2.21E−3 m, which can be calculated with the estimated value of the parameters (for example, $B = 5.0\text{E}{-2}$ m, $H = 6.0\text{E}{-2}$ m). In order to reduce the displacement amplitude, interval optimization method can be used. In this example we suppose the uncertainties of the design variables are specified in advance, that is, $\Delta B = 0.4\text{E}{-2}$ m, $\Delta H = 0.5\text{E}{-2}$ m.

There are two optimization parameters: B^c, H^c.

The interval optimization is to optimize the displacement amplitude at the x-direction of the node 10. The ranges of the optimization parameters B^c and H^c are 1.0E−2 m to 6.0E−2 m and 1.0E−2 m to 8.0E−2 m. Using Lagrange optimal algorithm, the approximate deterministic optimization can be solved. The results of the interval optimization are shown in Table 4. For comparison, the results of the deterministic optimization are also listed in Table 4.

From Table 4, it can be seen that the results of the deterministic optimization in 0.8754E−3 m, while the result of the interval optimization is an interval value [0.629E−3 m,1.278E−3 m].

Example 4. Consider the same frame structure shown in Fig. 3. In this example, we suppose the uncertainties of the design variables are also the optimization parameters. There are four optimization parameters: B^c, H^c, ΔB, ΔH.

The interval optimization is also to optimize the displacement amplitude at the x-direction of the node 10. The ranges of the optimization parameters B^c, H^c, ΔB, and ΔH are 1.0E−2 m to 6.0E−2 m, 1.0E−2 m to 8.0E−2 m, 0.3E−2 m to 0.6E−2 m, and 0.4E−2 to 0.8E−2 m.

Using the Lagrange optimal algorithm, the approximate deterministic optimization can be solved. The results of the interval optimization are shown in Table 5. For comparison, the results of the deterministic optimization are also listed in Table 5.

From Table 5, it can be seen that the displacement amplitude of the deterministic optimization is

Table 4
Comparison of results of deterministic optimization and interval optimization of the frame structure with two optimization parameters

X_i^c	Initial values	Deterministic optimization values	Interval optimization
			Mid-point values
B^c	5.5E−2	5.86E−2	5.8E−2
H^c	7.0E−2	7.86E−2	7.83E−2
$\Delta X_i(specified)$			Uncertainties
ΔB	0.4E−2	0.0	0.4E−2
ΔH	0.5E−2	0.0	0.5E−2
min $\Psi(\mathbf{X})$		0.8754E−3	[0.629E−3,1.278E−3]

Table 5
Comparison of results of deterministic optimization and interval optimization of the frame structure with four optimization parameters

X_i^c	Initial values	Deterministic optimization values	Interval optimization
			Mid-point values
B^c	5.5E−2	5.86E−2	5.7E−2
H^c	7.0E−2	7.86E−2	7.6E−2
ΔX_i			Uncertainties
ΔB	0.3E−2	0.0	0.2999E−2
ΔH	0.4E−2	0.0	0.3999E−2
min $\Psi(\mathbf{X})$		0.8754E−3	[0.6742E−3,1.057E−3]

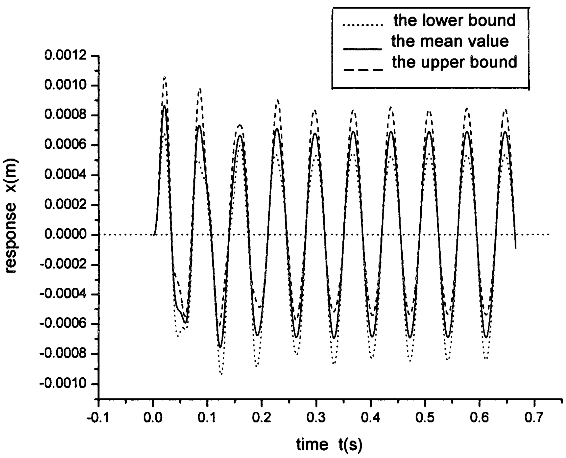

Fig. 4. The interval response at the optimal parameters.

0.8754E−3 m, while the result of the interval optimization is an interval value [0.6742E−3m,1.057E−3m].

From the Tables 4 and 5 it can be seen that, because we take more variables as optimization parameters, the interval value of the objective function obtained by Example 4 is sharper than that obtained by the Example 3. The interval response at the optimal parameters ($B^c = 5.7\text{E}{-2}$ m, $H^c = 7.6\text{E}{-2}$ m, $\Delta B = 0.2999\text{E}{-2}$ m, and $\Delta H = 0.3999\text{E}{-2}$ m) is given in Fig. 4.

8. Conclusions

In this paper, a new interval optimization method is proposed for vibration responses of structures with interval parameters. The interval optimization problem is transformed into the approximate deterministic optimization one, so we can use the standard algorithm for nonlinear optimization to solve the interval optimization problem. It can be seen that, using the interval optimization method, more information for the optimal structures can be obtained, such as how the optimization results change if the uncertainties of structural parameters are imposed on the structures. The conclusions are supported by the numerical examples. Because the present approach is based on the first-order Taylor expansion, the application of the approach is limited to the cases where the interval uncertainties of the interval parameters are small. If the interval uncertainties of the interval parameters are fairly large, in order to obtain higher computing accuracy, the second-order Taylor expansion should be considered.

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