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What is mathematical fuzzy logic ☆

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Abstract

The paper comments on the development and present state of fuzzy logic as a kind (branch) of mathematical logic. It is meant just as a contribution to the discussion on what fuzzy logic is, not as a systematic presentation of mathematical fuzzy logic.

Keywords: Fuzzy logic; Mathematical logic; Triangular forms

1. Origin, motivation, task

We mention some milestones of the development of fuzzy logic understood as a branch of mathematical (symbolic) logic.

Clearly, the story begins by Zadeh's first paper [40] on fuzzy sets. The term "fuzzy logic" is not used; but Zadeh mentions Kleene three-valued logic (just in passing).

Goguen's 1968–1969 paper [17] speaks on *logic of inexact concepts* but the term "fuzzy logic" occurs there (on p. 359; is this the first occurrence of the term in the literature?) The paper is very general, introduces algebras called *closg*, very near to algebras presently called residuated lattices, as algebras of truth functions of connectives for many-valued logics of inexact concepts, As an example he presents the unit real interval [0, 1] with product and its residuum (Goguen implication), thus a particular t-norm algebra (not speaking on t-norms). Zadeh has written several papers on fuzzy logic; an early paper is his "Fuzzy logic and approximate reasoning" [41] from 1975 (reprinted in [30]), where he uses connectives of Łukasiewicz logic min, max, $1 - x$, Łukasiewicz implication—but *not* strong conjunction. Note that what we call Łukasiewicz strong (or bold) conjunction or Łukasiewicz t-norm (the t-norm whose residuum is Łukasiewicz implication) was never explicitly used by Łukasiewicz. The first explicit use of this conjunction in the context of Zadeh's fuzzy logic appears to be the paper [16] by Giles. [1]

In Zadeh's understanding, fuzzy logic *uses* some many-valued logic but works with fuzzy truth values and his linguistic variables. Zadeh (and the majority of researchers up to today, including the present author) understands fuzzy logic as *truth functional*, i.e. having some truth functions for connectives determining the truth value of a compound

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[1] But Giles refers to papers by Chang and Klaua from mid-sixties using the bold conjunction in studying some generalized set theories over Łukasiewicz or similar logics.

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formula constructed using a connective uniquely from the truth values of the formula(s) to which the connective is applied. This makes fuzzy logic different from any probability theory (probability is evidently not truth functional). A general, not necessarily truth-functional approach is possible and was first investigated by Pavelka [36]; but Pavelka also developed in the quoted series of papers a truth-functional fuzzy logic which can be shown to be just Łukasiewicz ([0, 1]-valued) logic extended by truth constant $\bar{r}$ for each $r \in [0, 1]$. This approach has been continuously further developed by V. Novák. For a modern treatment of a non-truth-functional fuzzy logic see [15].

Triangular norms (t-norms) enter the game in early 80's of the last century. Dubois and Prade [8] wrote: "It is now well-known that a good model of a fuzzy set-theoretic intersection, or equivalently of a conjunction function in multivalued logics, is a triangular norm" and they refer to [1,7,38], also to papers from 1980. Gottwald presents in his 1984 German book [18] left-continuous t-norms and their residua, referring to Pedrycz [37]. Since then t-norms become "standard" semantics of fuzzy conjunction (cf. e.g. the 1995 monograph [29]).

At this place it is proper to make the distinction between *fuzzy logic in broad and narrow sense* (the term was coined by Zadeh), the former being a discipline using the notion of fuzzy logical connectives and other notions of the theory of fuzzy sets to develop methods of a sort of applied "approximate reasoning" (fuzzy IF-THEN rules, fuzzy controllers, fuzzy clustering and more or less anything else), whereas the latter discipline (fuzzy logic in the narrow sense) develops deductive systems of fuzzy logic as a many-valued logic with a comparative notion of truth very much in the style of classical mathematical logic (propositional and predicate calculi; axiomatization, (in)completeness, complexity, etc.).

It seems reasonable to call this kind of fuzzy logic (in the narrow sense) just *mathematical fuzzy logic*. The basic monograph on it is Hájek's [21] from 1998; Gottwald's Treatise on many-valued logic [19] contains a part devoted to (mathematical) fuzzy logic; the monograph [33] is a book on (mathematical) fuzzy logic stressing the approach using truth constants in the language. I also mention Turunen's [39]. There is a vast number of papers contributing to this topic; the references of this paper are by far incomplete.

What is the *motivation and the task* of mathematical fuzzy logic? Inspired by the "fuzzy logic in broad sense" (which is often developed by non-logicians) to construct symbolic logical calculi that can serve as foundation for the "broad-sense"-methods and, moreover, are meaningful as logics of inference under vagueness (or, as Goguen said, logics of inexact concepts). The reader is invited to take this as our (imprecise, informal) explanation of the term "mathematical fuzzy logic"; I offer no formal definition. Apparently systems that are t-*norm based* are of central importance; but this should be understood in a general sense, open to many alternatives. In the next two sections I survey a notion of t-norm based mathematical fuzzy logic with its double semantics—*standard* (with algebras of truth functions given by some t-norms) and *general* (working with algebras of truth functions taken from varieties generated by some standard algebras).

2. t-norm based fuzzy propositional logic

We quickly survey main facts; for a recent detailed survey see [20].

2.1. Continuous t-norms, basic fuzzy predicate logic

As commonly known, a t-norm is a binary operation $*$ on the real unit interval [0, 1] which is commutative, associative, non-decreasing in each argument and has 0 and 1 as zero and unit element. Its residuum is a binary operation on [0, 1] defined as $x \Rightarrow y = \max\{z \mid x * z \leqslant y\}$. A t-norm has residuum iff it is left continuous. Structure of *continuous* t-*norms* is well understood: roughly, each continuous t-norm is "constructed from copies of" three most important continuous t-norms called Łukasiewicz, Gödel and product t-norms. (No analogous characterization is known for left continuous t-norms.) A t-*algebra* $[0, 1]_*$ given by a continuous t-norm $*$ is the algebra $([0, 1], *, \Rightarrow, 0, 1)$ with two binary operations and two constants. Defined operations are $(-)x = x \Rightarrow 0$, $x \cap y = x * (x \Rightarrow y)$, $x \cup y = ((x \Rightarrow y) \Rightarrow y) \cap ((y \Rightarrow x) \Rightarrow x)$.[2] For each continuous t-norm $*$, $x \cap y$ is $\min(x, y)$ and $x \cup y$ is $\max(x, y)$.

This leads to a propositional calculus with starting connectives & (conjunction), $\rightarrow$ (implication) and truth constants $\bar{0}, \bar{1}$. Defined connectives are $\wedge, \vee, \neg$ (possibly others). Any t-algebra $[0, 1]_*$ ($*$ continuous) serves as the algebra of truth functions of connectives, which gives the notion of $*$-*tautologies* (formulas having $*$-value 1 for any

[2] The last identity is called Dummett's identity.

evaluation of propositional variables by truth values from [0, 1]). There is a simple *axiom system* defining (together with the deduction rule *modus ponens*) the *basic fuzzy propositional logic* BL (see [21]) which satisfies *standard completeness theorem*: a formula φ is provable in BL iff for each continuous t-norm $*$, φ is a $*$-tautology (proved in [4]). Formulas satisfying the latter condition are called standard BL-tautologies. Axioms for $*$-tautologies for a particular $*$ are got as an extension of axioms of BL and for some t-norms (Łukasiewicz, Gödel, product) have a very simple form.

The reader may ask if there is any natural interpretation of truth-functional semantics of connectives in fuzzy logic: if the truth degree of φ, ψ is x, y, respectively, what is the truth degree of $\varphi \& \psi$ computed as $x * y$? There is no definitive answer to this question but there are remarkable interpretations by Paris [34,35] for Gödel, product and Łukasiewicz and by Cicalese and Mundici [3] for BL using multi-channel Rényi-Ulam games (generalizing previous Mundici's interpretation of Łukasiewicz logic using Ulam games with lies).

Up to now we have dealt with *standard semantics,* having the unit real interval [0, 1] for the set of truth degrees and some continuous t-norms algebra $[0, 1]_*$ for the algebra of truth functions of connectives. Let $\mathcal{K}$ be a non-empty set of continuous t-norms (think of the set of all continuous t-norms or a singleton containing your favourite t-norm). A standard $\mathcal{K}$-tautology is of course a formula being a $*$-tautology for each $*$ in $\mathcal{K}$.

Now let $Var(\mathcal{K})$ be the variety of algebras $\mathbf{A} = (A, *, \Rightarrow, 0_A, 1_A)$ generated by all t-algebras $[0, 1]_*$ for $*$ in $\mathcal{K}$. It is easy to see that $\mathbf{A} \in Var(\mathcal{K})$ iff each standard $\mathcal{K}$-tautology is an $\mathbf{A}$-tautology (has identically the value 1_A if computed using $\mathbf{A}$ as the algebra of truth functions). Thus defining a general $\mathcal{K}$-tautology as a formula which is an $\mathbf{A}$-tautology for each $\mathbf{A} \in \mathcal{K}$ we immediately get that general $\mathcal{K}$-tautologies coincide with standard $\mathcal{K}$-tautologies. What one has to know is an equivalent algebraical characterization of elements of $Var(\mathcal{K})$ (general $\mathcal{K}$-algebras). For BL (i.e. $\mathcal{K}$ consisting of all continuous t-norms) we get the class of BL-algebras, i.e. residuated lattices $\mathbf{A} = (A, *, \Rightarrow, 0, 1, \cap, \cup)$ where the operations $\cap$ and $\cup$ are defined from $*, \Rightarrow$ as above, and $\mathbf{A}$ is a prelinear residuated lattice, i.e. $(A, \cap, \cup, 0, 1)$ is a lattice with extremal elements 0, 1, $(A, *, 1)$ is a commutative monoid, $\Rightarrow$ is the residuum of $*$ (i.e. for each x, y, z, $z \leqslant x \Rightarrow y$ iff $x * z \leqslant y$) and the prelinearity says that for all x, y, $(x \Rightarrow y) \cup (y \Rightarrow x) = 1$.

Clearly, not each BL-algebra is linearly ordered (think of the direct product of two t-algebras); but each BL-algebra is a subalgebra of the direct product of some linearly ordered BL-algebras (of BL-chains). This is called the subdirect representation property. It is important that we are not obliged to work with linearly ordered systems of truth degrees; but admitting partial order we do not change the logic.

General semantics for particular t-norms is well understood; in particular, for Łukasiewicz it is formed by the variety of MV-algebras, for Gödel G-algebras (prelinear Heyting algebras) and for product by so-called product algebras (see [21,19,20] and references thereof). Note that to keep this paper short, I do not discuss questions of strong completeness (semantic characterization of provability of formulas in theories over a fuzzy logic); see the references just mentioned. Similarly for results on computational complexity.

2.2. *Generalizing BL*

The advantage of the logic of continuous t-norms include definability of lattice connectives from conjunction and implication, known structure of continuous t-norms, well-known particular cases (Łukasiewicz, Gödel, product). Clearly the implication is a logical connective of utmost interest and to have the residuated implication (which has excellent logical properties) we have to work with left-continuous t-norms, thus (full) continuity is not necessary. In their pioneering paper [10] Esteva and Godo elaborated the logic MTL (monoidal t-norm fuzzy logic) and the corresponding variety of MTL-algebras (just giving up the condition $x \cap y = x * (x \Rightarrow y)$ and the corresponding axiom of BL, whereas adding some natural axioms on $\wedge$). Their logic is very much analogous to BL, but is the logic of left continuous t-norms in the same meaning as BL is the logic of continuous t-norms (this follows from the results of [28]). Stronger logics IMTL and ΠMTL generalizing Łukasiewicz and product logic were defined and showed to be logics of some classes of left continuous t-norms (an analogous generalization of Gödel logic gives just Gödel logic itself) [11,27].

Let us stress that in MTL-algebras starting operations are $*, \Rightarrow$ and $\cap$ (since $\cap$ is not definable from $*$ and $\Rightarrow$), thus the starting connectives of MTL are $\&, \Rightarrow, \wedge$ (and truth constants $\bar{0}, \bar{1}$); the variety of MTL-algebras is generated by t-norm algebras $[0, 1]_*$ where $*$ is a left-continuous t-norm and the operations are $*, \Rightarrow, \cap$; constants are 0 and 1. The union (supremum) is defined by the same formula as above. Also note at this occasion that in the references given the class of algebras forming the general semantics of a logic (BL, Ł, ..., MTL, ...) is not defined as a variety generated by some set of t-norm algebras but as a class of residuated lattices satisfying some conditions (guaranteeing soundness w.r.t. a given axiomatization of the logic); showing that it is just the variety generated by the respective set of t-norms then becomes a result on standard completeness (saying that standard tautologies coincide with general tautologies). MTL-algebras also have subdirect representation property.

It is rather interesting to investigate fragments of our logics resulting by restricting the language. First one may study *falsity free* fragments of BL, MTL etc. just deleting $\bar{0}$ from the language of the logic and deleting the constant 0 from the language of the corresponding algebras. The t-norms remain as they have been; but the variety generated by the respective reduct of t-norm algebras may (and does) contain algebras that are not just reducts of algebras from the variety with the original richer language. To be concrete: let $\mathcal{K}$ be the variety of all continuous t-norms. Consider the variety generated by the set of algebras $([0,1], *, \Rightarrow, 1)$ with $*$ in $\mathcal{K}$. You get the variety of *basic hoops* [12]; the corresponding algebraic definition is as follows:

A *hoop* is an algebra $\mathbf{A} = (A, *, \Rightarrow, 1)$ such that $*$ is a binary commutative operation with the unit element 1 and $\Rightarrow$ is a binary operation satisfying, for each x, y, z,

$$x \Rightarrow x = 1, \quad x * (x \Rightarrow y) = y * (y \Rightarrow x), \quad (x * y) \Rightarrow z = x \Rightarrow (y \Rightarrow z).$$

It follows that $*$ is associative. Order $x \leqslant y$ is defined as $x \Rightarrow y = 1$. This makes a hoop to a lattice that may and may not have a least element. A hoop is *basic* if it satisfies, for all x, y, z,

$$((x \Rightarrow y) \Rightarrow z) * ((y \Rightarrow x) \Rightarrow z) \leqslant z.$$

There is a very useful representation of BL-chains using basic hoop chains. The axioms of the hoop logic result from axioms of BL just deleting the only axiom containing $\bar{0}$, namely $\bar{0} \to \varphi$ (ex falso quodlibet). Since hoop logic does not have $\bar{0}$, it does not have negation; it is just the continuous t-norm logic without negation (and $\bar{0}$).

Similarly one can investigate the positive part of MTL; the corresponding algebras are *semihoops* (see [12]). The language of semihoops has starting connectives $\&, \to, \wedge$ (and $\bar{\wedge}$). One may go still further and consider t-norm algebras given by left-continuous t-norm and using only $*, \Rightarrow, 1$. The corresponding variety is the variety of basic quasihoops [24]; in it the definition $x \leqslant y$ iff $x \Rightarrow y \geqslant 1$ gives only a quasiorder (preorder).

For logics of the non-commutative (variants of) left-continuous t-norms see [22,23]. Let us also mention logics based on two t-norms (Łukasiewicz and product—the logic ŁΠ, see e.g. [9]), extensions of the described logics by *hedges*, notably by the so-called Baaz Delta [2] as well as logics with two negations [13].

2.3. Adding truth constants

As mentioned above, Pavelka in his [36] investigated a logic that turns out to be just Łukasiewicz logic with its standard semantics extended by truth constants $\bar{r}$ for each $r \in [0,1]$ ($\bar{r}$ has just the truth value r). Later I suggested to use only $\bar{r}$ for *rational* $r \in [0,1]$ which keeps the language countable and preserves Pavelka's results. V. Novák investigates this logic as a *logic with evaluated syntax* $\mathrm{Ev}_{Ł}$ (see [32]). Relying on Łukasiewicz logic is explained by the fact that Łukasiewicz t-norm is the only one whose residuum is continuous and consequently admits a certain (Pavelka-style) completeness theorem for theories over this logic. (Since I do not discuss theories I do not go into details; but we meet Pavelka-style completeness in the section on predicate logic.) Here we comment on extending Ł, G, Π by rational truth constants and the corresponding varieties. Thus let $\mathcal{L}$ be any of the just mentioned logics. $R\mathcal{L}$ is the extension of the language of $\mathcal{L}$ by truth constant $\bar{r}$ for each rational $r \in [0,1]$ and of the axioms of $\mathcal{L}$ by the following bookkeeping axioms:

$$(\bar{r}\&\bar{s}) \equiv (\overline{r * s}), (\bar{r} \to \bar{s}) \equiv (\overline{r \Rightarrow s}).$$

Let $[0,1]_{R\mathcal{L}}$ be the corresponding t-norm algebra, now expanded by names $\bar{r}$ of all rational elements of [0, 1]. Let $Var([0,1]_{R\mathcal{L}})$ be the corresponding variety. For $\mathcal{L}$ being Ł, G, Π we have *standard completeness*: For each formula φ, φ is provable (has a proof) in $R\mathcal{L}$ iff φ is a $[0,1]_{R\mathcal{L}}$-tautology. This is equivalent to the statement that $Var([0,1]_{R\mathcal{L}})$ is just the class of all $R\mathcal{L}$-algebras, i.e. for Ł all MV-algebras with constants $\bar{r}$ for rational $r \in [0,1]$ satisfying the bookkeeping axioms, similarly for G (G-algebras) and Π (product algebras). For Ł the result is in [21]; for G and Π the results are very new (see [14,5]). (For theories over $R\mathcal{L}$ the situation is different, details are omitted here).

3. Fuzzy t-norm based predicate logics

Admittedly propositional logics are important and interesting but it is only predicate logic (logic with quantifiers and structured atomic formulas) which is most important (and possibly most interesting). Let $\mathcal{L}$ be any of our propositional logics investigated till now; given by a set $\mathcal{K}$ of (left) continuous t-norms one can build a predicate logic $\mathcal{L}\forall$ with "classical" quantifiers $\forall, \exists$ over it. We shall quickly survey main facts; note that a detailed survey paper on fuzzy predicate logics [6] is under preparation. Let a predicate language (predicates with arities, object constants) be given.

A standard interpretation of our language is a structure

$$\mathbf{M} = (M, (R_P)_{P\,\text{predicate}}, (m_c)_{c\,\text{constant}})$$

with a non-empty domain M, where for each n-ary P, $R_P : M^n \to [0, 1]$ (n-ary fuzzy relation) and $m_c \in M$ for each constant c. Given a t-norm $*$ from $\mathcal{K}$, the value $\|\varphi\|^*_{\mathbf{M},v}$ of a formula φ in $\mathbf{M}$ (given by $*$ and an evaluation v of object variables by elements of M) is defined in the obvious (Tarski-style) way, for connectives using the operations given by $*$, and defining the truth value of a formula beginning by $\forall$ or $\exists$ as the infimum/supremum of the values of its instances. (See e.g. [21] for details.) A formula φ is a (standard) *$*$-tautology* if $\|\varphi\|^*_{\mathbf{M},v} = 1$ for each $\mathbf{M}$ and v; φ is a *standard $\mathcal{K}$-tautology* if it is a $*$-tautology for each $*$ in $\mathcal{K}$. This is the standard semantics of the predicate logic $\mathcal{L}\forall$. To define the *general semantics* we define, for each $\mathbf{A} \in Var(\mathcal{K})$, an $\mathbf{A}$-interpretation in the same way as above but R_P being $\mathbf{A}$-fuzzy relations, i.e. $R_P : M^n \to A$. The definition of $\|\varphi\|^{\mathbf{A}}_{\mathbf{M},v}$ is obvious, but need not be total since the suprema/infima in question need not exist. The interpretation is *safe* ($\mathbf{A}$-safe) if $\|\varphi\|^{\mathbf{A}}_{\mathbf{M},v}$ is defined for each φ and v.

A formula φ is an $\mathbf{A}$-tautology if $\|\varphi\|^{\mathbf{A}}_{\mathbf{M},v} = 1$ for each safe $\mathbf{A}$-interpretation $\mathbf{A}$ and each v; φ is a *general $\mathcal{L}\forall$-tautology*, if it is an $\mathbf{A}$-tautology for each *linearly ordered* $\mathbf{A} \in Var(\mathcal{K})$; φ is a *general $\mathcal{L}\forall^-$-tautology* if it is an $\mathbf{A}$-tautology for each $\mathbf{A} \in Var(\mathcal{K})$. The *axioms* of $\mathcal{L}\forall$ are the axioms of $\mathcal{L}$ (built from predicate formulas) plus five axioms for quantifiers: two substitution axioms: $(\forall x)\varphi(x) \to \varphi(t)$, $\varphi(t) \to (\exists x)\varphi(x)$ (for t substitutable for x in φ), two axioms on moving quantifiers over implication: $(\forall x)(\chi \to \varphi) \to (\chi \to (\forall x)\varphi)$, $(\forall x)(\varphi \to \chi) \to ((\exists x)\varphi \to \chi)$ (x not free in χ) and last axiom $(\forall 3) : (\forall x)(\chi \vee \varphi) \to (\chi \vee (\forall x)\varphi)$ (x not free in χ). Deduction rules are modus ponens and generalization.

The *strong completeness theorem* (valid for each of our discussed logics $\mathcal{L}\forall$) says that for each theory T over $\mathcal{L}\forall$ and each formula φ, T proves φ over $\mathcal{L}\forall$ if φ is true in each $\mathbf{A}$-model of T, for each *linearly ordered* $\mathbf{A} \in Var(\mathcal{K})$; an $\mathbf{A}$-model of T being a safe interpretation $\mathbf{M}$ in which all elements of T have the $\mathbf{A}$-value $\|\varphi\|^{\mathbf{A}}_{\mathbf{M},v} = 1$ (for all v). The logic $\mathcal{L}\forall^-$ results from $\mathcal{L}\forall$ by deleting the last axiom $(\forall 3)$; the strong completeness for theories over $\mathcal{L}\forall^-$ results by replacing $\mathcal{L}\forall$ by $\mathcal{L}\forall^-$ and deleting the restriction "linearly ordered", thus taking models over all $\mathbf{A} \in Var(\mathcal{K})$. For some logics the proofs are fully analogous to the proofs for BL$\forall$ (basic fuzzy predicate logic see [21]), for some other ones additional work is necessary; e.g. for the logics with non-commutative conjunction [26] and also for the logic of (basic) quasihoops (in which the disjunction $\vee$ is not available, see [24]), one has to formulate the axiom $(\forall 3)$ using only & and $\to$ (which turns out to be possible).

Thus general semantics behaves well; what about standard semantics? For Gödel logic G$\forall$ the standard tautologies coincide with general tautologies and we have the standard strong completeness (provability in a theory over G$\forall$ is equivalent to truth in all $[0, 1]_G$-models), but this is the only continuous t-norm with standard completeness. For each other continuous t-norm $*$, standard $*$-tautologies form a proper subset of the set of general $*$-tautologies and the former set is much more undecidable than the latter, e.g. the set of standard tautologies of BL$\forall$ is not arithmetical in the sense of arithmetical hierarchy (cf. the survey paper [25]). Surprisingly enough, for MTL$\forall$ we do have standard completeness, see [31].

For predicate Łukasiewicz logic with rational truth constants (alias Rational Pavelka predicate logic) we have a result called Pavelka completeness: given a theory T over RŁ$\forall$, define the *T-provability degree* of φ to be $|\varphi|_T = \sup\{r | T \vdash \bar{r} \to \varphi\}$ and the *T-truth degree* of φ to be $\|\varphi\|_T = \inf\{\|\varphi\|_{\mathbf{M}} | \mathbf{M}$ a $[0, 1]_{R\text{Ł}}$-model of $T\}$. The theorem says that $|\varphi|_T = \|\varphi\|_T$ (provability degree equals truth degree). But RŁ$\forall$ does not have standard completeness theorem (since even Ł$\forall$ does not have it). A formula φ is a standard RŁ$\forall$-tautology iff for each $r < 1$, RŁ$\forall$ proves $\bar{r} \to \varphi$ (saying that the truth value of φ is at least r) but it can happen that there is no proof of φ itself in RŁ$\forall$. Pavelka completeness *fails* for any continuous t-norm $*$ different from Łukasiewicz due to the fact that the corresponding residuum is not continuous. (This was shown by Pavelka himself in his paper.)

4. Conclusion

We have sketched some basic facts on t-norm based fuzzy logics, both propositional and predicate logics. We consider these logics to be of central importance for mathematical fuzzy logics as discussed above. They are t-norm based, i.e. their standard semantics is given by (left) continuous t-norms and their residua, but are general, i.e. develop their general semantics for which the algebras of truth functions of connectives are taken from the variety generated by the corresponding standard algebras. The interplay of both semantics shows the richness of these logics: general semantic is completely axiomatizable and is a well understood deductive system, whereas the standard semantics exhibits surprising (and beautiful) features of non-axiomatizability; the axiomatization of the general semantic is of course *sound* for the standard semantics, thus it is a powerful deductive system also for it. Mathematical t-norm based fuzzy logic understood in the described way is both a very interesting field of research in "pure" (symbolic, formal) logic and a very useful means of foundational analysis of methods of the fuzzy logic in broad sense.

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