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Deriving priorities from fuzzy pairwise comparison judgements

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Abstract

A new approach for deriving priorities from fuzzy pairwise comparison judgements is proposed, based on α-cuts decomposition of the fuzzy judgements into a series of interval comparisons. The assessment of the priorities from the pairwise comparison intervals is formulated as an optimisation problem, maximising the decision-maker's satisfaction with a specific crisp priority vector. A fuzzy preference programming method, which transforms the interval prioritisation task into a fuzzy linear programming problem is applied to derive optimal crisp priorities. Aggregating the optimal priorities, which correspond to different α-cut levels enables overall crisp scores of the prioritisation elements to be obtained.

A modification of the linear fuzzy preference programming method is also proposed to derive priorities directly from fuzzy judgements, without applying α-cut transformations. The formulation of the prioritisation problem as an optimisation task is similar to the previous approach, but it requires the solution of a non-linear optimisation program. The second approach also derives crisp priorities and has the advantage that it does not need additional aggregation and ranking procedures.

Both proposed methods are illustrated by numerical examples and compared to some of the existing fuzzy prioritisation methods.

Keywords: Fuzzy mathematical programming; Multiple criteria decision-making; Analytic hierarchy process; Fuzzy and interval comparisons

1. Introduction

The analytic hierarchy process (AHP) method [17] is widely used for multicriteria decision-making and has successfully been applied to many practical decision-making problems. In spite of its

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popularity, the method is often criticised for its inability to adequately handle the inherent uncertainty and imprecision associated with the mapping of a decision-maker's perception to crisp numbers [10]. The empirical effectiveness and theoretical validity of the AHP have also been discussed by many authors [3,5,12], and this discussion has focused on four main areas: the axiomatic foundation, the correct meaning of priorities, the 1–9 measurement scale and the rank reversal problem. However, most of the problems in these areas have been partially resolved, at least for three-level hierarchic structures [15].

It is not our intention to contribute further to that discussion. Rather, the main objective of this paper is to propose a new approach to tackling uncertainty and imprecision within the prioritisation process in the AHP, in particular, when the decision-maker's judgements are represented as fuzzy numbers or fuzzy sets.

In the AHP, the decision problem is structured hierarchically at different levels, each level consisting of a finite number of elements. The relative importance of the decision elements (i.e. the weights of the criteria and the scores of the alternatives) is assessed indirectly from comparison judgements.

Pairwise comparisons in the AHP assume that the decision-maker can compare any two elements E_i, E_j at the same level of the hierarchy and provide a numerical value a_{ij} for the ratio of their importance. If the element E_i is preferred to E_j then $a_{ij} > 1$. Correspondingly, the reciprocal property $a_{ji} = 1/a_{ij}$, $a_{ij} > 0$, for $j = 1, 2, \dots, n$, $i = 1, 2, \dots, n$, always holds.

Each set of comparisons for a level with n elements requires $n(n-1)/2$ judgements, which are further used to construct a positive reciprocal matrix of pairwise comparisons $A = \{a_{ij}\} \in \Re^{n \times n}$.

The priority vector $w = (w_1, w_2, \dots, w_n)^T$ may be obtained from the comparison matrix by applying some prioritisation method, e.g. the eigenvalue method, the logarithmic least squares method, the weighted least squares method and the goal programming method [2,17], or the fuzzy programming method [16] recently proposed by the author of this paper.

However, in many cases the preference model of the human decision-maker is uncertain and fuzzy and it is relatively difficult crisp numerical values of the comparison ratios to be provided. The decision-maker may be uncertain about his level of preference due to incomplete information or knowledge, inherent complexity and uncertainty within the decision environment, lack of an appropriate measure or scale.

A natural way to cope with uncertain judgements is to express the comparison ratios as fuzzy sets or fuzzy numbers, which incorporate the vagueness of the human thinking. When comparing two elements E_i and E_j, the exact numerical ratio a_{ij} can be approximated with a fuzzy ratio "about a_{ij}", which is represented by a fuzzy number $\tilde{a}_{ij}$.

Van Laarhoven and Pedrycz [14] extend the AHP to a fuzzy hierarchical analysis, using comparison matrices with triangular fuzzy numbers. They obtain fuzzy priorities $\tilde{w}_i$, $i = 1, 2, \dots, n$ by applying a fuzzy version of the Logarithmic least squares method.

Wagenknecht and Hartmann [18] employ the least squares method to calculate fuzzy priorities $\tilde{w}_i$, $i = 1, 2, \dots, n$, which approximate the fuzzy ratios $\tilde{a}_{ij}$ so that $\tilde{a}_{ij} \approx \tilde{w}_i / \tilde{w}_j$. The obtained fuzzy priorities are represented as $(L–R)$ fuzzy numbers. In the same paper, the authors propose another approach where the fuzzy priorities are calculated from a fuzzy comparison matrix using the geometric mean:

$$\tilde{w}_i = \prod_{j=1}^{n} (\tilde{a}_{ij})^{1/n}.$$

The approach proposed by Buckley [8], is rather similar to the second approach of Wagenknecht and Hartmann. Buckley employs trapezoidal fuzzy numbers, claiming that such numbers are more easily understood by experts. The prioritisation process is also based on the geometric mean, and the derived priorities are combined in the Saaty hierarchy to compute the final fuzzy scores, which are then compared by fuzzy ranking.

Another approach for fuzzy prioritisation, called synthetic extent analysis, is given in [9]. The author applies a simple arithmetic mean algorithm to find fuzzy priorities from comparison matrices, whose elements are represented by fuzzy triangular numbers. However, the arithmetic mean is a very naïve prioritisation approach, as shown by Saaty [17], and can be used only if the comparison matrices are consistent.

The fuzzy prioritisation methods mentioned above have some common characteristics. Firstly, they derive priorities from fuzzy comparison *matrices*. However, the approach of constructing fuzzy reciprocal matrices, taken by analogy from the crisp prioritisation methods leads to some problems, as demonstrated in the next section. In addition, in some cases the decision-maker might be unwilling or unable to provide all fuzzy comparisons necessary to construct full comparison matrices.

Secondly, all these methods derive *fuzzy* priorities and, after aggregating, the final scores of the alternatives are also represented as fuzzy numbers or fuzzy sets. Due to the large number of multiplication and addition operations, the resulting fuzzy scores have wide supports and overlap over a large range. As shown in [6,13], the normalisation procedure used in some of these methods may even result in irrational final fuzzy scores, where the *normalised upper value* < *normalised mean value* < *normalised lower value*.

Finally, the fuzzy prioritisation methods mentioned above require an additional *fuzzy ranking procedure* in order to compare the final fuzzy scores. The different ranking procedures, however, often give different ranking results [7].

In this paper, a new approach for deriving priorities from fuzzy comparison judgements is proposed, that eliminates some of the drawbacks of the existing fuzzy prioritisation methods. This approach does not require the construction of fuzzy comparison matrices and it can derive priorities from an incomplete set of fuzzy judgements. The proposed approach is also invariant to the specific form of the fuzzy sets used to represent the judgements, and can be applied when some of the judgements are represented as intervals or crisp values.

By using α-cuts, the initial fuzzy judgements are transformed into a series of interval judgements. A new fuzzy preference programming (FPP) method is employed to derive *crisp* priorities from the interval judgements, corresponding to each α-cut level, thus eliminating the need for an additional fuzzy ranking procedure. A simple aggregation is then used to obtain crisp overall values of the priorities.

Finally, a non-linear modification of the FPP method is proposed for a direct assessment of priorities without decomposing the fuzzy judgements by α-cuts.

2. Fuzzy comparison matrices

In this paper and in the examples, we represent the decision-maker's uncertain judgements using a specific form of normal fuzzy sets, which are called *fuzzy numbers* [11]; however, the specific form of the fuzzy sets does not restrict the applicability of our approach.

A normal fuzzy set $\tilde{N}$ is a triangular fuzzy number, defined by three real numbers $a \leqslant b \leqslant c$, and has a linear piecewise continuous membership function $\mu_{\tilde{N}}(x)$ with the following characteristics [11]:

1. a continuous mapping from $\Re$ to the closed interval [0,1];
2. $\mu_{\tilde{N}}(x) = 0$ for all $x \in [-\infty, a]$ and for all $x \in [c, +\infty]$;
3. strictly linearly increasing on $[a, b]$ and strictly linearly decreasing on $[b, c]$;
4. $\mu_{\tilde{N}}(x) = 1$ for $x = b$.

The fuzzy number $\tilde{N}$ can be expressed as a triple (a, b, c), where b is the most possible value of the fuzzy number; and a and c are the lower and the upper bounds, respectively, representing the scope of the fuzziness of the fuzzy number.

Let us consider a prioritisation problem with n unknown priorities $w = (w_1, w_2, \dots, w_n)^{\mathrm{T}}$, where the pairwise comparison judgements are represented by fuzzy numbers $\tilde{a}_{ij} = (l_{ij}, m_{ij}, u_{ij})$.

The known fuzzy prioritisation methods require a full set of $m = n(n-1)/2$ comparison judgements, in order to construct a positive reciprocal matrix of pairwise comparisons $\tilde{A} = \{\tilde{a}_{ij}\}$ of the type:

$$\tilde{A} = \begin{bmatrix} 1 & \tilde{a}_{12} & \cdots & \tilde{a}_{1n} \\ \tilde{a}_{21} & 1 & \cdots & \tilde{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{n1} & \tilde{a}_{n2} & \cdots & 1 \end{bmatrix}, \tag{1}$$

where $\tilde{a}_{ji} = 1/\tilde{a}_{ij} = (1/u_{ij}, 1/m_{ij}, 1/l_{ij})$. For definitions of the basic arithmetic operations on fuzzy numbers the reader could refer to [11].

This approach of constructing fuzzy reciprocal matrices, taken by analogy from the crisp prioritisation methods however leads to some problems, due to the non-linearity of the Saaty 1–9 scale in the region of values between 1/9 and 1. We will illustrate this using two simple examples.

Example 1. Consider a perfectly consistent two-dimensional prioritisation problem, where there are only two elements to be compared pairwisely. The decision-maker regards the first element as approximately two times more important than the second one, so his judgement can be represented by a symmetrical fuzzy number "about two", $\tilde{a}_{12} = (1, 2, 3)$. The solution of the prioritisation problem by existing fuzzy methods requires finding fuzzy priorities $\tilde{w}_1 > 0$ and $\tilde{w}_2 > 0$, such that their ratio approximately satisfies the initial judgement, i.e. $\tilde{r}_{12} = \tilde{w}_1/\tilde{w}_2 \approx \tilde{a}_{12}$. The normalisation constraint $\tilde{w}_1 + \tilde{w}_2 = \tilde{1}$ must also be satisfied.

In order to apply some of the existing fuzzy prioritisation methods we have to construct a fuzzy comparison matrix (1), calculating the reciprocal fuzzy judgement, which is $\tilde{a}_{21} = (1/3, 1/2, 1)$. It is evident, however, that the reciprocal fuzzy number is non-symmetrical. This asymmetry of the reciprocal judgements in the fuzzy comparison matrix, constructed from symmetrical elements, leads to non-symmetrical final fuzzy scores. For instance, applying the fuzzy geometric mean method [8], the following fuzzy normalised priorities can be obtained:

$$\tilde{w}_1 = (0.367, 0.667, 1.098), \quad \tilde{w}_2 = (0.210, 0.333, 0.634).$$

The resulting fuzzy ratio, corresponding to these fuzzy priorities can easily be calculated, using the division operator, defined in [11]:

$$\tilde{r}_{12} = \tilde{w}_1/\tilde{w}_2 = (0.577, 2.0, 5.196).$$

It is seen that the ratio $\tilde{r}_{12}$ is rather dissimilar to the initial fuzzy judgement $\tilde{a}_{12} = (1, 2, 3)$. Furthermore, the fuzzy ratio is strongly skewed, in contrast to the initial symmetrical judgement.

The resulting fuzzy priorities can be compared, using some of the fuzzy ranking methods. For example, the centre of gravity (CoG) defuzzification method [7] gives the following crisp values of the priorities:

$$w_{1c} = 0.71,\ w_{2c} = 0.393,\ \text{where the resulting crisp ratio is } r_{12c} = w_{1c}/w_{2c} = 1.808.$$

Solving the same two-dimensional problem with crisp judgements $a_{12} = 2$ and $a_{21} = 0.5$ by any crisp prioritisation method will give us perfectly consistent normalised priorities $w_{1n} = 2/3$, $w_{2n} = 1/3$, so that the ratio $r_{12n} = w_{1n}/w_{2n} = 2$ is equal to the initial judgement.

It can be concluded that the introduction of additional reciprocal elements and the construction of full fuzzy comparison matrices lead to some inaccuracy in the final results. Moreover, the skewed reciprocals might reverse the final ranking of the elements if an inverse ratio scale is used. It is proved by Barzilai [2] that the geometric mean method is independent of scale inversion and preserves rank strongly for crisp comparison matrices. But this property of the geometric mean does not hold in the fuzzy cases, even if the pairwise comparisons are perfectly consistent.

To illustrate the rank reversal phenomenon, let us consider again the simple two-dimensional problem, described in the previous example.

Example 2. Suppose that the decision-maker is almost indifferent between the two elements. If he is asked to assess by how much the first element is better than the second one, the answer can be expressed by the fuzzy number "about one", i.e. $\tilde{a}_{12} = (0.5, 1, 1.5)$. The corresponding reciprocal element in the fuzzy comparison matrix then is $\tilde{a}_{21} = (2/3, 1, 2)$, and the solution of the problem by the fuzzy geometric mean method gives the following normalised weights:

$$\tilde{w}_1 = (0.268, 0.5, 0.804), \quad \tilde{w}_2 = (0.309, 0.5, 0.928).$$

The resulting fuzzy priorities are shown in Fig. 1(a). Using the CoG ranking procedure, we can get the corresponding defuzzified crisp values

$$w_{1c} = 0.52 \quad \text{and} \quad w_{2c} = 0.58, \quad \text{where } r_{12c} = w_{1c}/w_{2c} = 0.9.$$

But if the decision-maker was asked by how much the second element is better or worse than the first one, he would provide the same fuzzy ratio "about one", i.e. $\tilde{a}_{21} = (0.5, 1, 1.5)$. In this case, the resulting comparison matrix is componentwisely inverse to the previous one. Solving the inverse fuzzy prioritisation problem, we can get

$$\tilde{w}_1 = (0.309, 0.5, 0.928), \quad \tilde{w}_2 = (0.268, 0.5, 0.804), \quad w_{1c} = 0.58, \quad w_{2c} = 0.52, \quad r_{12c} = 1.1.$$

Fig. 1(b) illustrates these results. It is evident that the ranking of the elements is reversed, hence the final solution depends on the description of the problem and it is not independent of the scale inversion.

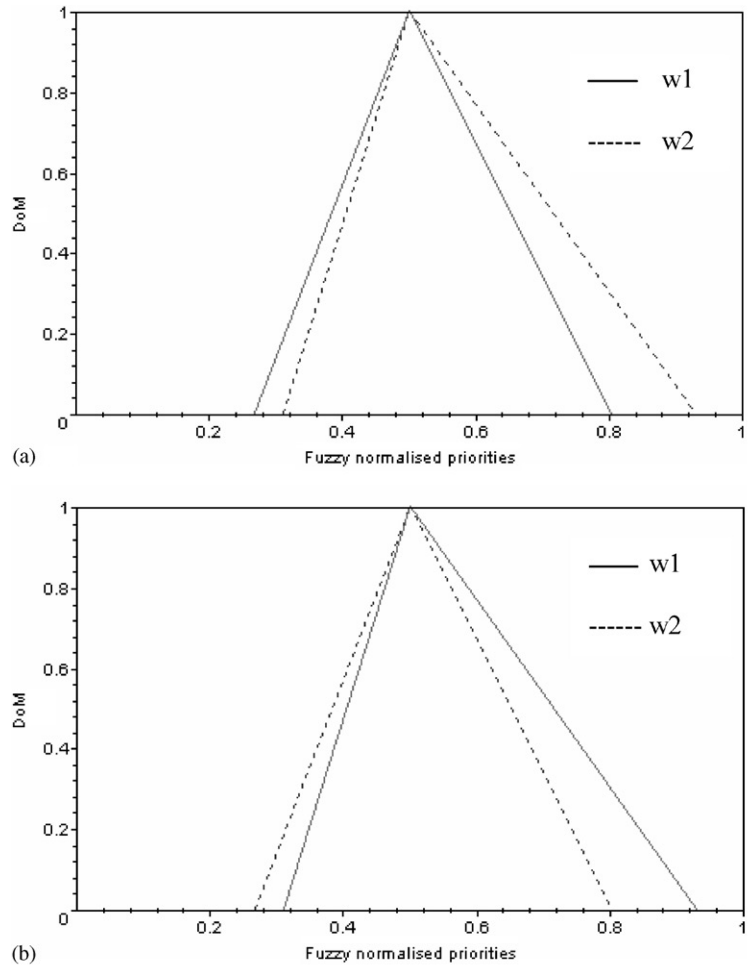

Fig. 1. Fuzzy solutions for Example 2: (a) $\tilde{a}_{12} = (0.5, 1, 1.5)$, (b) $\tilde{a}_{21} = (0.5, 1, 1.5)$.

Obviously, the violation of the strong rank reversal property of the fuzzy geometric mean method results from the non-linearity of the ratio scale and construction of skewed fuzzy reciprocal matrices. On the other hand, the reciprocal elements in these matrices do not provide some additional information, so their use should be avoided.

The fuzzy prioritisation approach, described in the following section does not require the construction of reciprocal matrices and thus could eliminate some of the problems discussed above.

3. Fuzzy prioritisation approach

Consider a prioritisation problem with n elements, and suppose that the decision-maker can provide a set $F = \{\tilde{a}_{ij}\}$ of $m \leqslant n(n-1)/2$ fuzzy comparison judgements, $i = 1, 2, \ldots, n-1$, $j = 2, 3, \ldots, n$, $j > i$, which are represented as normal convex fuzzy sets or fuzzy numbers. The crisp sets of the ratios between the unknown crisp priorities $w_i/w_j \in \Re^+$ that belong to the fuzzy judgement $\tilde{a}_{ij}$ to degree of α are called α-*level sets* (or simply α-cuts) of $\tilde{a}_{ij}$, and are defined as $a_{ij}(\alpha) = \{w_i/w_j \in \Re^+ \mid \mu_{\tilde{a}_{ij}}(w_i/w_j) \geqslant \alpha\}$ [20]. Using this concept, each fuzzy judgement $\tilde{a}_{ij}$ can be represented as a sequence of sets $a_{ij}(\alpha_l)$, $l = 1, 2, \ldots, L$, where $0 = \alpha_1 < \alpha_2 < \cdots < \alpha_L = 1$. For $\alpha = 0$, the corresponding α-level interval represents the *support* of the fuzzy judgement $\tilde{a}_{ij}$, while $a_{ij}(1)$ is the *core* of $\tilde{a}_{ij}$.

Since the fuzzy judgements are normal convex fuzzy sets or triangular fuzzy numbers $\tilde{a}_{ij} = (l_{ij}, m_{ij}, u_{ij})$, the α-level sets $a_{ij}(\alpha_l) = [l_{ij}(\alpha_l), u_{ij}(\alpha_l)]$ are closed intervals, such that $a_{ij}(1) \subseteq a_{ij}(\alpha_{L-1}) \subseteq \cdots \subseteq a_{ij}(0)$, where $l_{ij}(\alpha_l)$ and $u_{ij}(\alpha_l)$ are the lower and the upper bounds of the corresponding intervals. By applying α-cuts, the initial set of fuzzy comparisons $F = \{\tilde{a}_{ij}\}$ can be converted into a series of L interval sets $F_l = \{a_{ij}(\alpha_l)\}$, $l = 1, 2, \ldots, L$.

The main idea behind the proposed approach for fuzzy prioritisation is to find crisp values of the priorities $w(\alpha_l) = (w_1(\alpha_l), w_2(\alpha_l), \ldots, w_n(\alpha_l))^{\mathrm{T}}$, $l = 1, 2, \ldots, L$, corresponding to each interval set of pairwise comparisons F_l and then to aggregate the results in order to obtain final crisp values of the priorities.

The fuzzy programming method proposed by Mikhailov [16] that derives priorities from crisp comparison matrices is modified to find crisp priorities from the interval sets of judgements. The new FPP method, described in the next section represents the interval judgements at each α-level as fuzzy linear constraints and defines a convex fuzzy feasible area of all possible solutions. Then the assessment of the priorities is stated as an optimisation problem, maximising the decision-maker's satisfaction with a specific crisp priority vector.

4. Fuzzy preference programming method

Consider a set of $m \leqslant n(n-1)/2$ interval pairwise comparison judgements $F_l = \{l_{ij}(\alpha_l), u_{ij}(\alpha_l)\}$ at the level $\alpha = \alpha_l$. When the interval judgements are consistent, there are many priority vectors, whose elements satisfy the inequalities

$$l_{ij}(\alpha) \leqslant \frac{w_i}{w_j} \leqslant u_{ij}(\alpha). \tag{2}$$

If the judgements are inconsistent, there is no priority vector that satisfies all interval judgements simultaneously. However, it is reasonable to try and find a vector that satisfies all judgements "as well as possible". This implies that a good enough solution vector has to satisfy all interval judgements approximately, or

$$l_{ij}(\alpha) \tilde{\leqslant} \frac{w_i}{w_j} \tilde{\leqslant} u_{ij}(\alpha), \tag{3}$$

where the symbol $\tilde{\leqslant}$ denotes the statement "fuzzy less or equal to".

In order to handle the above inequalities easily we can represent them as a set of single-side fuzzy constraints:

$$\begin{aligned} & w_i - w_j u_{ij}(\alpha) \tilde{\leqslant} 0, \\ & -w_i + w_j l_{ij}(\alpha) \tilde{\leqslant} 0. \end{aligned} \tag{4}$$

The above set of $2m$ fuzzy constraints can be given in a matrix form as

$$Rw \tilde{\leqslant} 0, \tag{5}$$

where the matrix $R \in \Re^{2m\times n}$.

The kth row of (5), denoted by $R_k w \tilde{\leqslant} 0$, $k = 1, 2, \ldots, 2m$, represents a fuzzy linear constraint and may be characterised by a linear membership function of the type

$$\mu_k(R_k w) = \begin{cases} 1 - \frac{R_k w}{d_k}, & R_k w \leqslant d_k, \\ 0, & R_k w \geqslant d_k, \end{cases} \tag{6}$$

where d_k is a tolerance parameter, denoting the admissible interval of approximate satisfaction of the crisp inequality $R_k w \leqslant 0$.

The membership function (6) represents the decision-maker's satisfaction with a specific priority vector, with respect to the kth single-side constraint (4). The value of the of membership function $\mu_k(R_k w)$ is equal to zero when the corresponding crisp constraint $R_k w \leqslant 0$ is strongly violated; it linearly increases and takes positive values less than 1 when the constraint is approximately satisfied and is greater than 1 when the constraint is fully satisfied.

Let $\mu_k(R_k w)$, $k = 1, 2, \ldots, 2m$ be membership functions of the fuzzy constraints $R_k w \tilde{\leqslant} 0$ on the $(n-1)$-dimensional simplex

$$Q^{n-1} = \{(w_1, \ldots, w_n) \mid w_i > 0,\ w_1 + \cdots + w_n = 1\}. \tag{7}$$

Definition 1. The *fuzzy feasible area* $\tilde{P}$ on the simplex Q^{n-1} is a fuzzy set, described by the membership function

$$\mu_{\tilde{P}}(w) = [\min\{\mu_1(R_1 w), \ldots, \mu_m(R_m w)\} \mid w_1 + \cdots + w_n = 1]. \tag{8}$$

The fuzzy feasible area is defined as an intersection of all fuzzy constraints on the simplex. If the initial interval judgements are inconsistent, by choosing "large enough" tolerance parameters we can obtain a non-empty fuzzy feasible area. It can easily be proved that a non-empty feasible area $\tilde{P}$ on the simplex Q^{n-1} is a convex fuzzy set [16].

The convex fuzzy feasible area $\tilde{P}$ represents the overall satisfaction of the decision-maker with a specific crisp priority vector w. Assuming that the decision-maker is interested in the best-possible solution, it is reasonable to determine a priority vector that maximises his overall degree of satisfaction.

Definition 2. The *maximising solution* is a crisp vector w^*, which corresponds to the maximum of the fuzzy feasible area

$$\mu_{\tilde{P}}(w^*) = \max[\min\{\mu_1(R_1 w), \ldots, \mu_m(R_m w)\} \mid w_1 + \cdots + w_n = 1]. \tag{9}$$

Since the fuzzy feasible area $\tilde{P}$ is a convex set and all fuzzy constraints are defined as convex sets, there is always a point w^* on the simplex that has a maximum degree of membership in $\tilde{P}$.

The max–min operator for deriving a maximising solution was proposed by Bellman and Zadeh [4] for general decision-making problems with fuzzy goals and fuzzy constraints. Their approach integrates the goals and the constraints, so that the difference between them essentially disappears. Zimmermann [20] employs Bellman and Zadeh's idea for problems with linear fuzzy goals and linear fuzzy constraints and shows that the max–min fuzzy linear problem can be transformed into a conventional linear program.

Similarly, by introducing a new variable λ, measuring the degree of membership of a given priority vector in the fuzzy feasible area $\tilde{P}$ and using (6) and (9), we can represent the problem of finding the maximising solution as a linear program:

$$\begin{aligned} &\text{maximise} && \lambda \\ &\text{subject to} && d_k\lambda + R_k w \leqslant d_k, \\ & && \sum_{i=1}^{n} w_i = 1,\ w_i > 0,\ i = 1,2,\ldots,n,\ k = 1,2,\ldots,2m. \end{aligned} \tag{10}$$

The optimal solution to the above linear program is a vector (w^*, λ^*), whose first component represents the priority vector that has a maximum degree of membership in the fuzzy feasible area, whereas the second component gives the value of that maximum degree, $\lambda^* = \mu_{\tilde{P}}(w^*)$. The value of λ^* measures the degree of satisfaction and is a natural indicator for the inconsistency of the decision-maker's judgements, so we can call it a *consistency index* [16]. When the interval judgements are consistent, $\lambda^* \geqslant 1$. For inconsistent judgements the consistency index λ^* takes a value between 1 and 0 that depends on the degree of inconsistency and the values of the tolerance parameters d_k.

The tolerance parameters should be chosen large enough to ensure the non-emptiness of the feasible area $\tilde{P}$ and a positive value of λ^*. The next section shows that values of these parameters greater than or equal to 1 satisfy such requirements.

If the fuzzy judgements are symmetrical, it is reasonable for all tolerance parameters to be set equal to 1, since usually the decision maker has no preferences about his individual pairwise comparison judgements. It should be noted that equal values of all tolerance parameters do not affect the value of the maximising solution w^*.

A simple algorithm for adjusting the tolerance parameters, which maximises the degree of membership when the fuzzy judgements are non-symmetrical, is proposed in Section 6.

5. Numerical results

Let us consider the solution to the two-dimensional prioritisation problems, described in Section 2, by applying the proposed approach. The fuzzy judgement provided by the decision-maker is a triangular fuzzy number $\tilde{a}_{12} = (l_{12}, m_{12}, u_{12})$. The priority ratios at each α-cut level should satisfy $l_{12}(\alpha) \leqslant w_1/w_2 \leqslant u_{12}(\alpha)$, where the bounds of the α-cut intervals are

$$l_{12}(\alpha) = \alpha(m_{12} - l_{12}) + l_{12},$$

$$u_{12}(\alpha) = \alpha(m_{12} - u_{12}) + u_{12}.$$

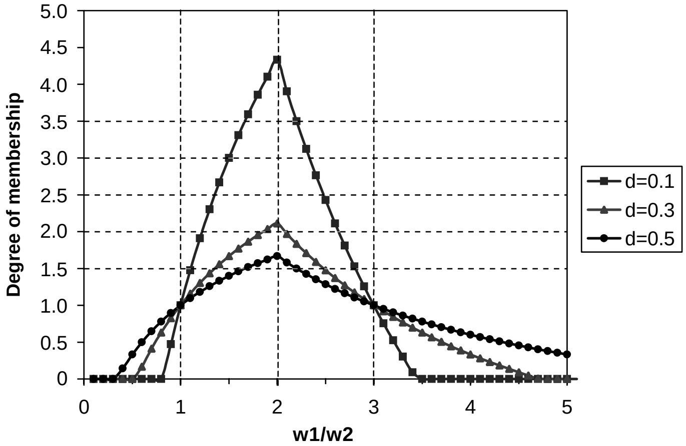

Fig. 2. The fuzzy feasible area (Example 1) as a function of the tolerance parameters.

The solution of the prioritisation problem at each α-cut level can be obtained by solving a linear program of the type:

$$\begin{aligned} \text{maximise} \quad & \lambda \\ \text{subject to} \quad & d_1\lambda + w_1 - u_{12}(\alpha)w_2 \leqslant d_1, \\ & d_2\lambda - w_1 + l_{12}(\alpha)w_2 \leqslant d_2, \\ & w_1 + w_2 = 1, \; w_i > 0, \quad i = 1, 2. \end{aligned} \tag{11}$$

If the tolerance parameters d_1 and d_2, corresponding to both interval bounds are equal, $d_1 = d_2 = d$, the solution of (11) does not depend on the specific value of d.

This can be illustrated, considering the solution of the linear program for $\alpha = 0$, when the fuzzy comparison judgement is $\tilde{a}_{12} = (1, 2, 3)$, as in Example 1. In this case, the interval bounds represent the support of the fuzzy number $\tilde{a}_{12}$, or we have $a_{12}(0) = (1, 3)$. The membership function of the fuzzy feasible area $\tilde{P}$ on the simplex $w_1 + w_2 = 1$ for different values of the tolerance parameter d is shown in Fig. 2.

It is evident that the value of the maximising solution ratio $w_1/w_2 = 2$ does not depend on the value of d and lies exactly in the middle of the interval $a_{12}(0) = (1, 3)$. The value of the consistency index λ (i.e. the maximum degree of membership), however, depends on d, but it is always greater than 1, indicating a consistent solution.

The projections of the fuzzy linear constraints (4) on the w_1–w_2 plane, for different values of the tolerance parameter d are shown in Fig. 3. It is evident that if $d = 1$, the scope of the feasible area $\tilde{P}$ is very large and encloses the whole simplex line $w_1 + w_2 = 1$. It follows that such a value of the tolerance parameters guarantees non-emptiness of the fuzzy feasible area, even if we have many inconsistent judgements. It should be mentioned that two-dimensional prioritisation problems are always consistent.

The solutions to the linear problem (11) for each α-cut level are equal, $w_1 = 0.667$, $w_2 = 0.333$, because the initial fuzzy judgement $\tilde{a}_{12} = (1, 2, 3)$ is symmetrical, and the means of all α-cut intervals

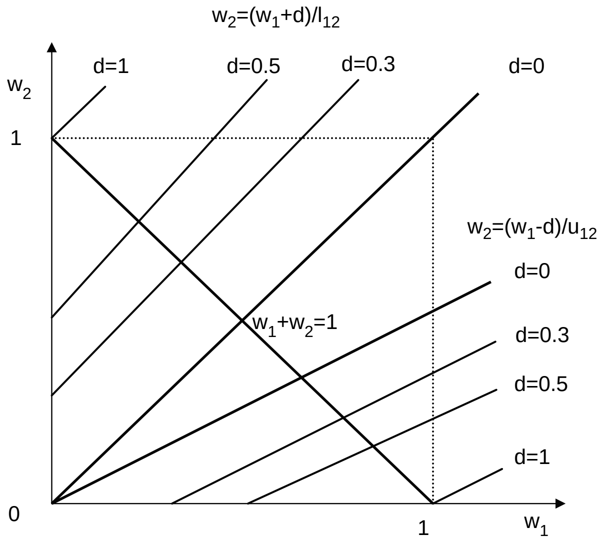

Fig. 3. Fuzzy linear constraints of the two-dimensional prioritisation problem.

are equal to the mean of the fuzzy judgement $m_{12}=2$. The optimal solution ratio $w_1/w_2=2$ corresponds to the mean of the fuzzy judgement and equally satisfies both interval boundary constraints. For each α-cut level, the value of λ is greater than 1, which indicates that we have a series of consistent prioritisation problems.

Similar results can be obtained, when $\tilde{a}_{12}=(0.5,1,1.5)$ (see Example 2 of Section 2). In this case, the solutions at each α-level are also equal, $w_1=0.5$, $w_2=0.5$, and the solution ratio lies at the mean of the fuzzy judgement.

Example 3. Consider again a two-dimensional fuzzy prioritisation problem. If the fuzzy judgement is not symmetrical, then the solutions to the linear programs at each α-level are not equal. For example, if $\tilde{a}_{12}=(0.5,1,3)$, then we have the solutions, given in Table 1. But it can be observed that the solution ratios are also exactly in the middle of the α-level intervals, as in the previous examples.

Example 4. In order to illustrate the performance of our approach, in particular when some of the interval judgements are inconsistent, we will consider the example given by Boender [6]. The decision-maker compares three elements and provides the following fuzzy judgements:

$$\tilde{a}_{21}=(2.5,3,3.5),\quad \tilde{a}_{31}=(4,5,6),\quad \tilde{a}_{32}=(1.5,2,2.5).$$

Setting all tolerance parameters equal to 1 and applying the proposed approach, we can get the results shown in Table 2 and Fig. 4.

It is seen that the value of the consistency index $\lambda>1$ for all $\alpha\leqslant 0.7$, which indicates that the corresponding interval judgements for these α-levels are consistent, i.e. the solution ratios lie within these intervals.

Table 1
Solutions of the FPP method for each α-cut level, Example 3

α	$l_{12}(\alpha)$	$u_{12}(\alpha)$	w_1	w_2	w_1/w_2	λ
0	0.5	3	0.636	0.364	1.750	1.455
0.1	0.55	2.8	0.626	0.374	1.675	1.421
0.2	0.6	2.6	0.615	0.385	1.600	1.385
0.3	0.65	2.4	0.604	0.396	1.525	1.347
0.4	0.7	2.2	0.592	0.408	1.450	1.306
0.5	0.75	2	0.579	0.421	1.375	1.263
0.6	0.8	1.8	0.565	0.435	1.300	1.217
0.7	0.85	1.6	0.551	0.449	1.225	1.169
0.8	0.9	1.4	0.535	0.465	1.150	1.116
0.9	0.95	1.2	0.518	0.482	1.075	1.060
1.0	1.0	1.0	0.500	0.500	1.000	1.000

Table 2
Solutions of the FPP method for each α-cut level, Example 4

α	w_1	w_2	w_3	λ
0	0.1111	0.3333	0.5556	1.056
0.1	0.1099	0.3297	0.5604	1.049
0.2	0.1087	0.3261	0.5652	1.043
0.3	0.1075	0.3226	0.5699	1.038
0.4	0.1075	0.3197	0.5728	1.029
0.5	0.1081	0.3171	0.5748	1.020
0.6	0.1087	0.3146	0.5767	1.010
0.7	0.1093	0.3123	0.5785	1.001
0.8	0.1099	0.3099	0.5802	0.9913
0.9	0.1105	0.3077	0.5818	0.9817
1.0	0.1111	0.3056	0.5833	0.9722

A graphical illustration of such consistent interval judgements at level $\alpha = 0.2$ and the corresponding optimal solution are shown in Fig. 5. The upper and lower constraints of each interval are represented as lines in the w_1–w_2 plane via elimination of the third dependent variable, $w_3 = 1 - w_1 - w_2$ in the constraint equations, as shown in [1].

Fig. 6 represents an inconsistent set of interval judgements at level $\alpha = 0.8$, where the intersection of all linear interval constraints is an empty set. The solution obtained by the FPP method satisfies approximately all interval judgements at that level.

The fuzzy solutions obtained by three other fuzzy methods [6] are represented in Table 3 for comparison. It is seen from Tables 2 and 3 that the range of our crisp solutions over all α-cut levels is much smaller.

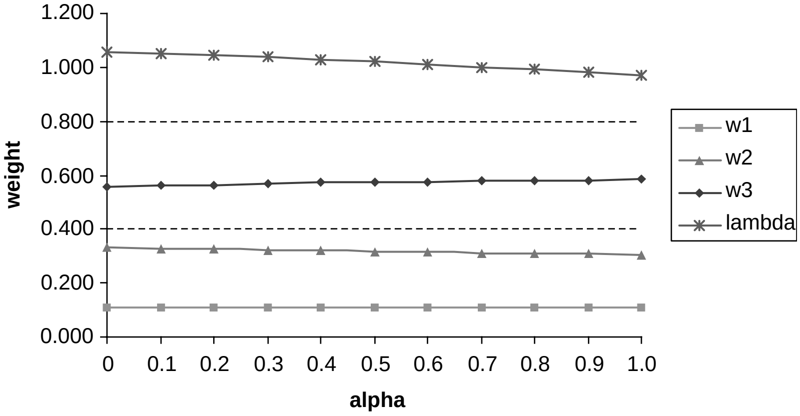

Fig. 4. Results for the three-dimensional Example 4.

Table 3
Fuzzy results for Example 4

	$\tilde{w}_1$	$\tilde{w}_2$	$\tilde{w}_3$
Buckley	(0.09, 0.11, 0.15)	(0.24, 0.31, 0.42)	(0.42, 0.58, 0.77)
Van Laarhoven, Pedrycz	(0.09, 0.11, 0.13)	(0.25, 0.31, 0.40)	(0.44, 0.58, 0.75)
Boender et al.	(0.10, 0.11, 0.12)	(0.28, 0.31, 0.35)	(0.50, 0.58, 0.66)

6. Adjustment of the tolerance parameters

From the previous examples it can be observed that in perfectly consistent cases, the solution ratios lie in the middle of the intervals, if the tolerance parameters are equal. Moreover, if the fuzzy judgements are symmetrical, the solution ratios are also equal to the means of these judgements. Generally, if the condition for perfect consistency $m_{ij} = m_{ik}m_{kj}$ holds and the fuzzy judgements are symmetrical, the solution ratios obtained by the FPP method, are always in the means m_{ij} of the judgements. Since the means have the highest degree of membership in the initial fuzzy judgements, it is reasonable to try to adjust the tolerance parameters in the membership functions (6) so that the solution ratios have the maximum degree of membership, when the judgements are non-symmetrical or inconsistent.

Consider a single interval $a_{ij}(\alpha) = [l_{ij}(\alpha), u_{ij}(\alpha)]$. The intersection of the positive parts of the linear membership functions (6), corresponding to its lower and upper bound defines a convex membership function for this interval $\mu_{ij} = \min(\mu_{ij}^{\mathrm{L}}, \mu_{ij}^{\mathrm{U}})$ of the type shown in Fig. 2, where

$$\mu_{ij}^{\mathrm{L}} = 1 - \frac{(-w_i + l_{ij}(\alpha)w_j)}{d_{ij}^{\mathrm{L}}} = 1 + \frac{w_i/w_j - l_{ij}(\alpha)}{d_{ij}^{\mathrm{L}}/w_j},$$

$$\mu_{ij}^{\mathrm{U}} = 1 - \frac{(w_i - u_{ij}(\alpha)w_j)}{d_{ij}^{\mathrm{U}}} = 1 + \frac{u_{ij}(\alpha) - w_i/w_j}{d_{ij}^{\mathrm{U}}/w_j}. \tag{12}$$

Here d_{ij}^{L} and d_{ij}^{U} denote the lower and upper tolerance parameters for this interval.

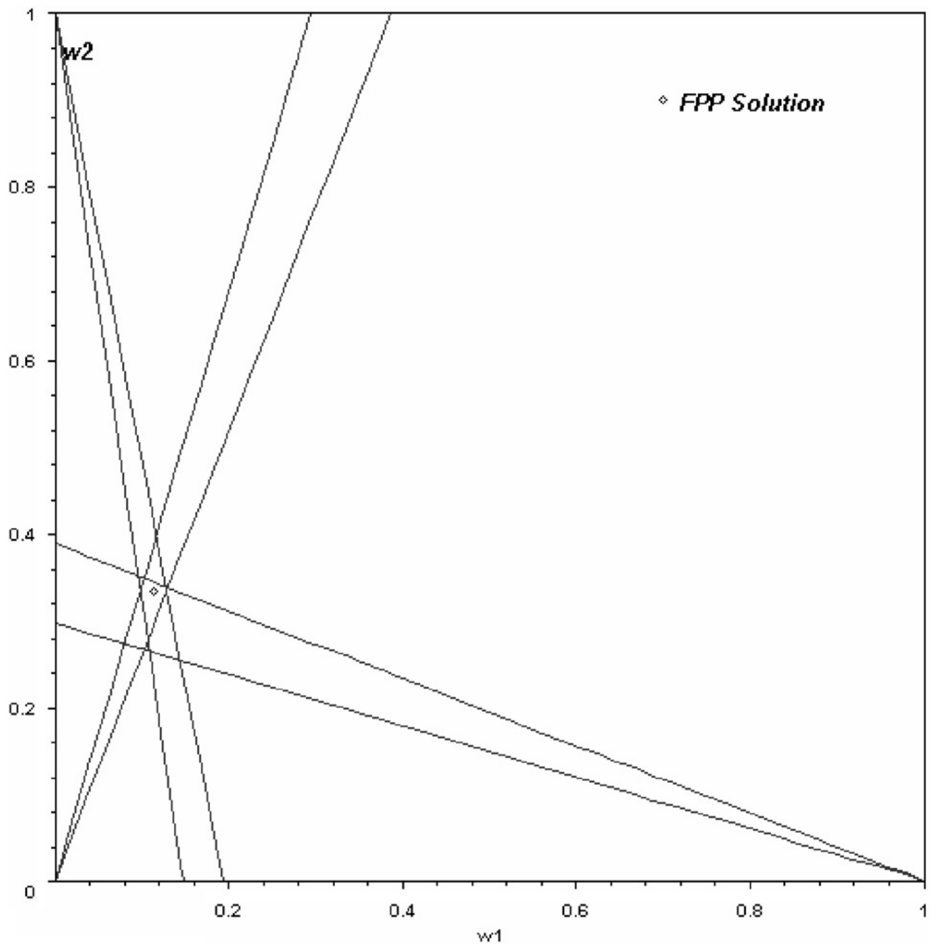

Fig. 5. Consistent interval judgements, $\alpha = 0.2$, Example 4.

The interval membership function $\mu_{ij} = \min(\mu_{ij}^{\mathrm{L}}, \mu_{ij}^{\mathrm{U}})$ has a maximum at the intersection point $\mu_{ij}^{\mathrm{L}} = \mu_{ij}^{\mathrm{U}}$, corresponding to a ratio that can be easily obtained from (12):

$$\frac{w_i}{w_j} = \frac{d_{ij}^{\mathrm{U}} l_{ij}(\alpha) + d_{ij}^{\mathrm{L}} u_{ij}(\alpha)}{d_{ij}^{\mathrm{U}} + d_{ij}^{\mathrm{L}}}. \tag{13}$$

When the tolerance parameters are equal, then the maximising ratio is in the middle of the interval:

$$\frac{w_i}{w_j} = \frac{l_{ij}(\alpha) + u_{ij}(\alpha)}{2} = c_{ij}(\alpha). \tag{14}$$

If the fuzzy judgement $\tilde{a}_{ij} = (l_{ij}, m_{ij}, u_{ij})$ is symmetrical, the projection of the mean m_{ij} on the α-level interval $a_{ij}(\alpha) = [l_{ij}(\alpha), u_{ij}(\alpha)]$ coincides with the midpoint of that interval $c_{ij}(\alpha)$. This indicates that by applying equal tolerance parameters, the maximising ratio will have a maximum

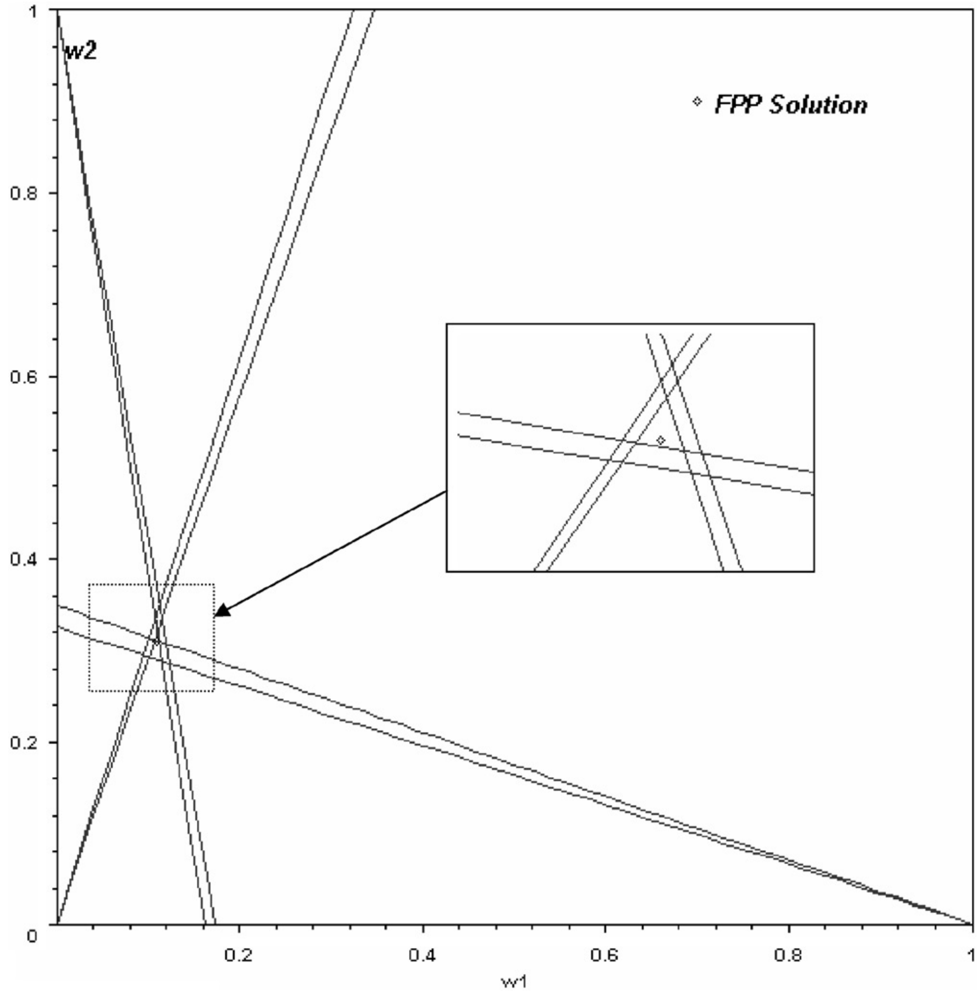

Fig. 6. Inconsistent interval judgements, $\alpha = 0.8$, Example 4.

degree in the membership function of the initial fuzzy judgment. But if the fuzzy judgement is not symmetrical, then the tolerance parameters could be chosen so that the maximising ratio (13) corresponds to the mean m_{ij}, therefore the ratio has a maximum degree of membership in the fuzzy set $\tilde{a}_{ij} = (l_{ij}, m_{ij}, u_{ij})$.

Substituting $w_i/w_j = m_{ij}$ in (13), and setting $\Delta^{L}(\alpha) = m_{ij} - l_{ij}(\alpha)$, $\Delta^{U}(\alpha) = u_{ij}(\alpha) - m_{ij}$, we have

$$\frac{\Delta^{L}(\alpha)}{d_{ij}^{L}} = \frac{\Delta^{U}(\alpha)}{d_{ij}^{U}}. \tag{15}$$

Eq. (15) can be used for an automatic adjustment of the tolerance parameters, in order to obtain a solution ratio maximising the degree of membership in the fuzzy judgements. Because we have two tolerance parameters for each interval, the parameter corresponding to the smaller deviation $\Delta^{L}(\alpha)$

or $\Delta^{\mathrm{U}}(\alpha)$ should be fixed, for example equal to 1. Then the second tolerance parameter, determined from (15) will have a value greater than 1, which guarantees a non-empty fuzzy feasible area. If the fuzzy judgement is symmetrical, then $\Delta^{\mathrm{L}}(\alpha)=\Delta^{\mathrm{U}}(\alpha)$, and it follows that $d_{ij}^{\mathrm{L}}=d_{ij}^{\mathrm{U}}=1$.

Applying the proposed procedure for adjustment of the tolerance parameters to the non-symmetrical two-dimensional problem from Example 3, at each α-level we obtain equal results, $w_1=0.5$, $w_2=0.5$, so all solution ratios are equal to the mean of the fuzzy judgement. Because the fuzzy judgements in the three-dimensional Example 4 are all symmetrical, there is no need for additional adjustment of the tolerance parameters.

7. Aggregation of the priorities

Decomposing the fuzzy comparison judgements into a series of interval judgements by α-cuts and applying the FPP method, we can obtain a sequence of crisp priorities, corresponding to each α-cut level

$$w(\alpha_l)=(w_1(\alpha_l),w_2(\alpha_l),\ldots,w_n(\alpha_l))^{\mathrm{T}},\quad l=1,\ldots,L,\ 0=\alpha_1<\alpha_2<\cdots<\alpha_L=1.$$

However, the relative importance of all priorities is not the same and depends on the level of α. A small value of α yields a construction of interval judgements, having large spreads, which indicates a high level of uncertainty and correspondingly, less reliable priorities. Basically, the support of the fuzzy judgements is the safest, but the most pessimistic bracketing of the unknown priority ratios [10,11]. A larger value of α yields smaller but more optimistic interval judgements, whose upper and lower bounds have greater degrees of membership in the initial fuzzy sets. When the fuzzy judgements are represented as fuzzy triangular numbers, the cutting at level $\alpha=1$ produces a set of crisp judgements, equal to the most possible value m_{ij} of the fuzzy comparison ratios $\tilde{a}_{ij}=(l_{ij},m_{ij},u_{ij})$. Hence, the decision-maker would put more trust in the priorities, corresponding to higher levels of α.

These considerations suggest that the value of α can be used as a weighting factor of the solutions, so we can obtain aggregated values of the priorities by a weighted sum of the type

$$W_j=\sum_{l=1}^{L}\alpha_l w_j(\alpha_l)\Bigg/\sum_{l=1}^{L}\alpha_l. \tag{16}$$

The aggregated values of the priorities for all examples from the previous sections are shown in Tables 4 and 5.

The proposed approach for prioritisation can be used to find global priorities within the AHP, but then the aggregation should be done after deriving the overall scores of all alternatives at each α-level cut.

Let us suppose that we have a specific three-level decision hierarchy with n criteria and m alternatives. We can obtain a set of fuzzy comparisons judgements $F=\{\tilde{a}_{ij}\}$, $i=1,2,\ldots,n-1$, $j=2,3,\ldots,n$, $j>i$, comparing pairwisely all criteria at the second level. For each criterion C_i, $i=1,\ldots,n$ we can form another set of fuzzy comparisons $S_i=\{\tilde{b}_{ij}\}$, $i=1,2,\ldots,m-1$, $j=2,3,\ldots,m$, $j>i$, evaluating the relative importance of the alternatives regarding this criterion.

Table 4
Results for all two-dimensional examples

	Linear method (aggregated priorities)				Non-linear method		
	Equal tolerance parameters		Adjusted tolerance parameters				
	W_1	W_2	W_1	W_2	w_1	w_2	λ
Example 1	0.667	0.333	0.667	0.333	0.667	0.333	1.0
Example 2	0.5	0.5	0.5	0.5	0.5	0.5	1.0
Example 3	0.548	0.452	0.5	0.5	0.5	0.5	1.0

Table 5
Results for the three-dimensional Example 4

Linear method (aggregated priorities)			Non-linear method			
W_1	W_2	W_3	w_1	w_2	w_3	λ
0.1095	0.3126	0.5779	0.1093	0.3121	0.5786	0.708

Decomposing all comparison judgements by α-cuts and applying the FPP method, described above, we can obtain a sequence of crisp weights $w(\alpha_l)=(w_1(\alpha_l),w_2(\alpha_l),\dots,w_n(\alpha_l))^{\mathrm{T}}$ and scores $s_j(\alpha_l)=(s_{j1}(\alpha_l),s_{j2}(\alpha_l),\dots,s_{jn}(\alpha_l))^{\mathrm{T}}$, $j=1,2,\dots,m$, $l=1,2,\dots,L$, which are then used in the aggregation procedure.

As in the original AHP method, the overall score of the jth alternative at each α-cut level can be calculated using the weighted sum

$$r_j(\alpha_l)=\sum_{i=1}^{n} s_{ji}(\alpha_l)w_i(\alpha_l). \tag{17}$$

The overall scores of the alternatives, derived in such a way are crisp values.

Definition 3. The alternative A_i strongly dominates the alternative A_j if and only if $r_i(\alpha_l)>r_j(\alpha_l)$ for all $l=1,2,\dots,L$.

The alternative that strongly dominates the others (if it exists) should be chosen in the final ranking. When none of the alternatives dominates for all α values, an additional ranking, as in (16) should be carried out.

The weighted overall score of the jth alternative can be defined as

$$R_j=\sum_{l=1}^{L}\alpha_l r_j(\alpha_l)\Big/\sum_{l=1}^{L}\alpha_l. \tag{18}$$

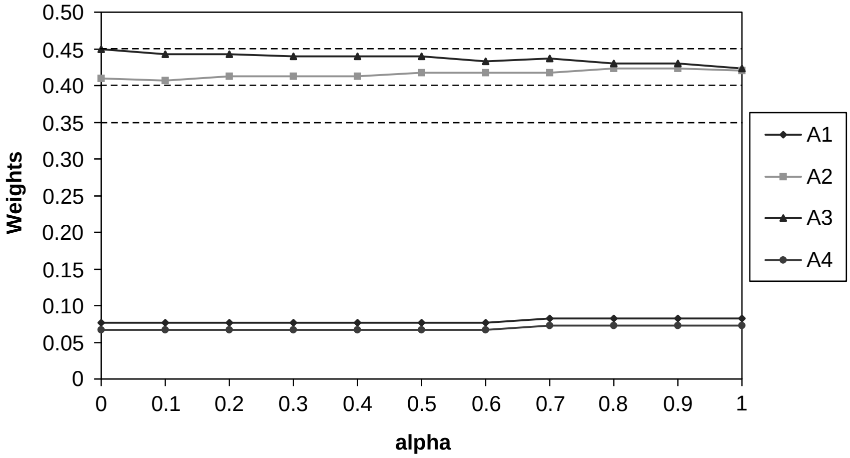

Fig. 7. Overall scores of the alternatives, Buckley's example.

8. Numerical example

We consider the example of Buckley [8], where the government wishes to rank various energy sources from most important to least important. The alternatives are $A_1 = $nuclear, $A_2 = $hydroelectric, $A_3 = $fossil, and $A_4 = $solar. There are two criteria. C1 represents economical and political considerations, while C_2 represents military and defence considerations. For the data of the pair-wise comparisons at each level of the hierarchy the reader can refer to [8]. It should be noted that the comparison ratios in this example are represented as fuzzy trapezoidal numbers.

The overall scores of the alternatives $r_j(\alpha_l)$ derived by our approach are shown graphically in Fig. 7. It can be seen that $r_3 > r_2 > r_1 > r_4$ for all values of α. According to our definition of strong dominance, we can conclude that A_3 is the most important energy source, followed by A_2, while A_4 is the least important one. A_1 is just slightly better than A_4.

Solving this example by the fuzzy geometric mean method, Buckley concludes that A_2 and A_3 have highest and approximately equal importance, while A_1 and A_4 are less important alternatives. Since the fuzzy scores of A_2 and A_3, and of those of A_1 and A_4, obtained by Buckley's fuzzy method overlap strongly, the author suggests that an additional study comparing only A_2 and A_3 would be needed if only one best alternative is required. In our approach, the overall scores of A_2 and A_3, and correspondingly those of A_1 and A_4 are also quite close, but the alternatives can be distinguished and ranked more precisely.

9. Non-linear fuzzy prioritisation

The proposed fuzzy approach to prioritisation needs a number of α-cuts, solving the linear programs (10), and eventually an aggregation of the priorities derived at the different α-levels. In order to avoid some of these steps, in this section we propose a non-linear method for prioritisation, which can find directly crisp values of priorities from a set of comparison judgements, represented as triangular fuzzy numbers.

We want to find a crisp priority vector, so that the ratios approximately satisfy the initial fuzzy judgements $\tilde{a}_{ij} = (l_{ij}, m_{ij}, u_{ij})$ or

$$l_{ij} \tilde{\leqslant} \frac{w_i}{w_j} \tilde{\leqslant} u_{ij}. \tag{19}$$

Instead of transforming these double-side inequalities into single-side linear fuzzy constraints, as in (4), for each fuzzy judgement we can construct a membership function, which is linear with respect to w_i/w_j:

$$\mu_{ij}\left(\frac{w_i}{w_j}\right) = \begin{cases} \dfrac{(\frac{w_i}{w_j} - l_{ij})}{m_{ij} - l_{ij}}, & \dfrac{w_i}{w_j} \leqslant m_{ij}, \\[2ex] \dfrac{(u_{ij} - \frac{w_i}{w_j})}{u_{ij} - m_{ij}}, & \dfrac{w_i}{w_j} \geqslant m_{ij}. \end{cases} \tag{20}$$

Function (20) is linearly increasing over the interval $(-\infty, m_{ij})$ and linearly decreasing over the interval (m_{ij}, ∞). It takes negative values when $w_i/w_j < l_{ij}$ or $w_i/w_j > u_{ij}$ and has a maximum value $\mu_{ij} = 1$ at $w_i/w_j = m_{ij}$. Over the range (l_{ij}, u_{ij}) the membership function (20) coincides with the fuzzy triangular judgment $\tilde{a}_{ij} = (l_{ij}, m_{ij}, u_{ij})$.

Compared to the membership functions (6), which are linear with respect to the priorities w_i, $i = 1, 2, \ldots, n$, but lead to a non-linear membership function of the feasible area, regarding the ratios w_i/w_j, as shown in Fig. 2, function (20) is non-linear with respect to the decision variables, but provides a fuzzy feasible area, linear in these ratios.

As in Section 4, we can define a fuzzy feasible area on simplex (7), as the intersection of all membership functions (20) and apply a max–min-approach for finding the maximising solution. This leads to the following non-linear optimisation problem:

$$\begin{aligned} &\text{maximise} && \lambda \\ &\text{subject to} && (m_{ij} - l_{ij})\lambda w_j - w_i + l_{ij} w_j \leqslant 0, \\ & && (u_{ij} - m_{ij})\lambda w_j + w_i - u_{ij} w_j \leqslant 0, \\ & && \sum_{k=1}^{n} w_k = 1, \ w_k > 0, \quad k = 1, 2, \ldots, n \\ & && i = 1, 2, \ldots, n-1, \ j = 2, 3, \ldots, n, \ j > i. \end{aligned} \tag{21}$$

In contrast to the previous optimisation problem (10), which is linear and can easily be solved using the simplex method, the solution of the above non-linear problem (21) needs some appropriate numerical method for non-linear optimisation to be employed. The optimal value of the consistency index λ, if it is positive, indicates that all solution ratios completely satisfy the initial judgments, i.e. $l_{ij} \leqslant w_i/w_j \leqslant u_{ij}$. If the consistency index has a negative value it indicates that the fuzzy judgements are strongly inconsistent and the solution ratios approximately satisfy them.

Modifying the membership functions (20) by introducing tolerance parameters as in (6), we can obtain an extended fuzzy feasible area, whose membership function is positive for inconsistent ratios as well. Then, in the inconsistent cases, the consistency index will also have a positive value.

The sequential quadratic programming method, implemented in the CFSQP package [19] was used for solving all previous examples by the proposed non-linear prioritisation method. The results are shown in Tables 4 and 5.

In the consistent two-dimensional examples with symmetrical fuzzy numbers, the priorities derived by the non-linear method are equal to those of the linear one, because the maximums of the linear and non-linear fuzzy feasible areas correspond to the same ratio, but if the fuzzy judgements are skewed or inconsistent, the results are slightly different.

10. Summary and conclusions

The linear and non-linear methods for solving the fuzzy prioritisation problem, proposed in this paper have some common characteristics. Both methods:

- derive priorities from fuzzy pairwise comparison judgements and do not need the construction of fuzzy comparison matrices of skewed reciprocal elements;
- allow for prioritisation from an incomplete set of judgements;
- use a max–min optimisation approach;
- derive crisp priorities and do not need an additional ranking procedure;
- can easily be applied for group decision-making.

The proposed α-cut approach has some further advantages. It treats all fuzzy judgements in a unique way and is invariant to the specific form or the shape of the fuzzy sets. This property provides opportunity for solving prioritisation problems with mixed types of comparison judgements, such as fuzzy sets of different forms, intervals or crisp numbers (singletons). Finally, the prioritisation problem is stated as a linear optimisation program, which can easily be solved.

The main advantage of the non-linear method for prioritisation is that it does not need an additional aggregation and ranking, but it requires a non-linear optimisation procedure. In its present formulation, the non-linear method is suitable for prioritisation problems, where the judgements are represented as triangular fuzzy sets, but it can easily be modified for other types of fuzzy judgements.

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