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# Interpretation of inequality constraints involving interval coefficients and a solution to interval linear programming

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**Abstract**

The modern trend in Operations Research methodology deserves modelling of all relevant vague or uncertain information involved in a real decision problem. Generally, vagueness is modelled by a fuzzy approach and uncertainty by a stochastic approach. In some cases, a decision maker may prefer using interval numbers as coefficients of an inexact relationship. As a coefficient an interval assumes an extent of tolerance or a region that the parameter can possibly take. However, its use in the optimization problems is not much attended as it merits.

This paper defines an interval linear programming problem as an extension of the classical linear programming problem to an inexact environment. On the basis of a comparative study on ordering interval numbers, inequality constraints involving interval coefficients are reduced in their satisfactory crisp equivalent forms and a satisfactory solution of the problem is defined. A numerical example is also given. © 2001 Elsevier Science B.V. All rights reserved.

*Keywords:* Interval number; Inequality relation; Mathematical programming

## 1. Introduction

In conventional mathematical programming, coefficients of problems are usually determined by the experts as crisp values. But in reality, in an imprecise and uncertain environment, it is an unrealistic assumption that the knowledge and representation of an expert are so precise. Hence, in order to develop good Operations Research methodology fuzzy and stochastic approaches are frequently used to describe and treat imprecise and uncertain elements present in a real decision problem. In fuzzy programming problems [2,6,10] the constraints and goals are viewed as fuzzy sets and it is assumed that their membership functions are known. On the other hand, in stochastic programming problems [1,5,9,12] the coefficients are viewed as random variables and it is also assumed that their probability distributions are known. These membership functions and probability distributions play important roles in their corresponding methods. However, in reality, to a decision maker (DM) it is not always easy to specify the membership function or the probability distribution in an inexact environment.

At least in some of the cases, use of an interval coefficient may serve the purpose better. Though by using $\alpha$-cuts, fuzzy numbers can be degenerated into interval numbers [13], deliberately we keep this concept out of the scope of this paper. Here, an interval number is

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considered as an extension of a real number and as a real subset of the real line $\Re$ [7]. As a coefficient an interval also signifies the extent of tolerance (or a region) that the parameter can possibly take. However, in decision problems its use is not much attended as its merits.

Let's refer [11] here a very good example of using interval numbers in an optimization problem:

> There are 1000 chickens raised in a chicken farm and they are raised with two kinds of forages – soya and millet. It is known that each chicken eats 1.000–1.130 kg of forage every day and that for good weight gain it needs at least 0.21–0.23 kg of protein and 0.004–0.006 kg of calcium everyday. Per kg of soya contains 48–52% protein and 0.3–0.8% calcium and its price is 0.38–0.42 Yuan. Per kg of millet contains 8.5–11.5% protein and 0.3% calcium and its price is 0.20 Yuan.
> How should the forage be mixed in order to minimize expense on forage?

Most of the parameters used in this problem are inexact and perhaps appropriately given in terms of simple intervals. In reality inexactness of this kind can be cited in countless numbers.

The optimization problem can be structured as follows:

$$\begin{aligned}
\text{Minimize} \quad & Z = [0.38, 0.42]x_1 + 0.20x_2 \\
\text{subject to} \quad & x_1 + x_2 = [1, 1.130] \times 1000, \\
& [0.48, 0.52]x_1 + [0.085, 0.115]x_2 \\
& \quad \geqslant [0.21, 0.23] \times 1000, \\
& [0.005, 0.008]x_1 + 0.003x_2 \\
& \quad \geqslant [0.004, 0.006] \times 1000, \\
& x_1, x_2 \geqslant 0.
\end{aligned}$$

However, for solution, techniques of classical linear programming cannot be applied if and unless the above interval-valued structure of the problem be reduced into a standard linear programming structure and for that we have to clear up the following main issues:

- First, regarding interpretation and realization of the inequality relations involving interval coefficients.
- Second, regarding interpretation and realization of the objective 'Min' with respect to an inexact environment.

In this paper, we concentrate on a satisfactory solution approach based on DM's interpretation of inequality relations and objective of the problem with respect to the inexact environment.

This paper is organized as follows. In Section 2 notations of interval number and the interval arithmetics are briefly explained. Section 3 along with its four subsections, give an elaborate study on inequality relation with interval coefficient in search of interpreting and realizing the relation as a constraint of an optimization problem defined in an inexact environment. Section 4 describes the solution principle of an interval linear programming problem, and solution to a previously cited problem [11] and efficiency of our methodology. Section 5 includes the concluding remarks and the future scope.

## 2. The basic interval arithmetic

All lower case letters denote real numbers and the upper case letters denote the interval numbers or the closed intervals on $\Re$.

*2.1.* $A = [a_L, a_R] = \{a\colon a_L \leqslant a \leqslant a_R, a \in \Re\}$, where $a_L$ and $a_R$ are left and right limit of the interval $A$ on the real line $\Re$, respectively. If $a_L = a_R$, then $A = [a, a]$ is a real number.

Interval $A$ is alternatively represented as $A = \langle m(A), w(A)\rangle$ where, $m(A)$ and $w(A)$ are the mid-point and half-width (or simply be termed as 'width') of interval $A$, i.e.,

$$m(A) = \tfrac{1}{2}(a_L + a_R), \qquad w(A) = \tfrac{1}{2}(a_R - a_L).$$

*2.2.* Let $* \in \{+, -, ., \div\}$ be a binary operation on the set of real numbers.

Then, $A \circledast B = \{a * b,\ a \in A,\ b \in B\}$ defines a binary operation on the set of closed intervals. In case of division it is assumed that $0 \notin B$.

If $\lambda$ is a scalar, then

$$\lambda.A = \lambda[a_L, a_R] = \begin{cases} \lambda[a_L, a_R] & \text{for } \lambda \geqslant 0, \\ \lambda[a_R, a_L] & \text{for } \lambda < 0. \end{cases}$$

The extended addition $\oplus$ and extended subtraction $\ominus$, are defined as follows:

$$A \oplus B = [a_L + b_L, a_R + b_R],$$

$$A \ominus B = [a_L - b_R, a_R - b_L].$$

The following equations also hold for $A \oplus B$ and $A \ominus B$:

$$m(A \oplus B) = m(A) + m(B),$$

$$m(A \ominus B) = m(A) - m(B),$$

$$w(A \oplus B) = w(A \ominus B) = w(A) + w(B).$$

## 3. Inequality relation with interval coefficients

An extensive research and wide coverage on interval arithmetics and its applications can be found in Moore [7]. Here we find two transitive order relations defined over intervals: the first one as an extension of '$<$' on the real line as

$$A < B \text{ iff } a_R < b_L$$

and the other as an extension of the concept of set inclusion i.e.

$$A \subseteq B \text{ iff } a_L \geqslant b_L \quad \text{and} \quad a_R \leqslant b_R.$$

These order relations cannot explain ranking between two overlapping intervals. The extension of the set inclusion here only describes the condition that the interval $A$ is nested in $B$; but it cannot order $A$ and $B$ in terms of value. We need to develop a definition of comparing two interval numbers.

Ishibuchi and Tanaka [4] approached the problem of ranking two interval numbers more prominently. In their approach, in a maximization problem if intervals $A$ and $B$ are two, say, profit intervals, then maximum of $A$ and $B$ can be defined by an order relation $\leqslant_{LR}$ between $A$ and $B$ as follows:

$$A \leqslant_{LR} B \text{ iff } a_L \leqslant b_L \quad \text{and} \quad a_R \leqslant b_R,$$

$$A <_{LR} B \text{ iff } A \leqslant_{LR} B \quad \text{and} \quad A \neq B.$$

Ishibuchi and Tanaka [4] suggested an another order relation $\leqslant_{mw}$ where, $\leqslant_{LR}$ cannot be applied, as follows:

$$A \leqslant_{mw} B \text{ iff } m(A) \leqslant m(B) \quad \text{and} \quad w(A) \geqslant w(B),$$

$$A <_{mw} B \text{ iff } A \leqslant_{mw} B \quad \text{and} \quad A \neq B.$$

Both of the above order relations $\leqslant_{LR}$ and $\leqslant_{mw}$ are antisymmetric, reflexive and transitive and hence, define partial ordering between intervals. Ishibuchi and Tanaka [4] showed that both of the order relations never conflict in the sense that there exists no such pair of $A$ and $B$ ($A \neq B$) so that $A \leqslant_{LR} B$ and $B \leqslant_{mw} A$ hold.

However, in a recent work Sengupta and Pal [8] showed that there exists a set of pairs of intervals for which both of $\leqslant_{LR}$ and $\leqslant_{mw}$ do not hold. They proposed a simple and efficient index for comparing any two interval numbers on the real line through decision maker's satisfaction.

### 3.1. The acceptability index

**Definition 3.1.1.** Let $\circledless$ be an extended order relation between the intervals $A = [a_L, a_R]$ and $B = [b_L, b_R]$ on the real line $\mathfrak{R}$, then for $m(A) \leqslant m(B)$, we construct a premise $A \circledless B$ which implies that $A$ is inferior to $B$ (or $B$ is superior to $A$). Here, the term 'inferior to' ('superior to') is analogous to 'less than' ('greater than').

**Definition 3.1.2.** Let $I$ be the set of all closed intervals on the real line $\mathfrak{R}$. Here, we further define an *acceptability function* $\mathscr{A} : I \times I \to [0, \infty)$ such that $\mathscr{A}(A \circledless B)$ or $\mathscr{A}_{\circledless}(A, B)$, or, in short,

$$\mathscr{A}_{\circledless} = \frac{(m(B) - m(A))}{(w(B) + w(A))},$$

where $w(B) + w(A) \neq 0$. $\mathscr{A}_{\circledless}$ may be interpreted as the *grade of acceptability of the 'first interval to be inferior to the second interval'*.

The grade of acceptability of $A \circledless B$ may be classified and interpreted further on the basis of comparative position of mean and width of interval $B$ with respect to those of interval $A$ as follows:

$$\mathscr{A}(A \lessdot B)\begin{cases} =0 & \text{if } m(A)=m(B),\\ >0,<1 & \text{if } m(A)<m(B) \\ & \quad \text{and } a_R>b_L,\\ \geqslant 1 & \text{if } m(A)<m(B) \\ & \quad \text{and } a_R \leqslant b_L. \end{cases}$$

If $\mathscr{A}(A \lessdot B)=0$, then the premise '$A$ is inferior to $B$' is not accepted. If $0<\mathscr{A}(A \lessdot B)<1$, then the interpreter accepts the premise $(A \lessdot B)$ with different grades of satisfaction ranging from zero to one (excluding zero and one). If $\mathscr{A}(A \lessdot B)\geqslant 1$, the interpreter is absolutely satisfied with the premise $(A \lessdot B)$ or in other words, he accepts that $(A \lessdot B)$ is true.

**Remark 3.1.3.** If $\mathscr{A}(A \lessdot B)>0$, then for a maximizing problem (say $A$ and $B$ are two alternative interval profits and the problem is to choose maximum profit), interval $B$ is preferred to $A$ and for a minimizing problem (say, $A$ and $B$ are two interval costs), $A$ is preferred to $B$ in terms of value.

**Remarks 3.1.4.** For any sort of value judgement the $\mathscr{A}$-index consistently satisfies the DM: For any two intervals $A$ and $B$ on $\mathfrak{R}$,

either $\mathscr{A}(A \lessdot B)>0$

or $\mathscr{A}(B \lessdot A)>0$

or $\mathscr{A}(A \lessdot B)=\mathscr{A}(B \lessdot A)=0$.

**Remark 3.1.5.** The proposed index is transitive; for any three intervals $A$, $B$ and $C$ on $\mathfrak{R}$.

if $\mathscr{A}(A \lessdot B)\geqslant 0$ and $\mathscr{A}(B \lessdot C)\geqslant 0$,

then $\mathscr{A}(A \lessdot C)\geqslant 0$.

But it *does not mean* that $\mathscr{A}(A \lessdot C)\geqslant \max(\mathscr{A}(A \lessdot B), \mathscr{A}(B \lessdot C))$.

**Proposition 3.1.6.** *Let* $\mathscr{A}(A \lessdot B)\geqslant 0$ *then there exists a family of intervals* $\{B_\lambda=\langle m(B_\lambda), w(B_\lambda)\rangle$: $\lambda$ *is real and* $\geqslant w(A)/(w(B)+w(A))\}$ *on* $\mathfrak{R}$ *for which*

$$\mathscr{A}(A \lessdot B_\lambda)=\mathscr{A}(A \lessdot B),$$

*where*

$$m(B_\lambda)=m(A)+\lambda(m(B)-m(A))$$

*and*

$$w(B_\lambda)=-w(A)+\lambda(w(B)+w(A)).$$

**Proof.** Let $B_1$ be an interval such that

$$\mathscr{A}(A \lessdot B_1)=\mathscr{A}(A \lessdot B)$$
$$\Rightarrow \frac{m(B_1)-m(A)}{w(B_1)+w(A)}=\frac{m(B)-m(A)}{w(B)+w(A)}$$
$$\Rightarrow \frac{m(B_1)-m(A)}{m(B)-m(A)}=\frac{w(B_1)+w(A)}{w(B)+w(A)}=\lambda \text{ say},$$

where $\lambda$ is any nonzero finite number.

Hence,

$$m(B_1)=m(A)+\lambda\{m(B)-m(A)\},$$
$$w(B_1)=-w(A)+\lambda\{w(B)+w(A)\}.$$

Since, $w(B_1)$ cannot be negative, $\lambda$ is to be restricted by $\lambda\geqslant w(A)/(w(A)+w(B))$.

As the interval $B_1$ depends on $\lambda$, denoting it by $B_\lambda$ we get the theorem.

**Remark 3.1.7.** If $\lambda_1<\lambda_2$, then $\mathscr{A}(B_{\lambda_1} \lessdot B_{\lambda_2})>0$.

**Remark 3.1.8.** If $\mathscr{A}(A \lessdot B)>0$, then there does not exist any $B'\in\{B_\lambda\}$ for which $\mathscr{A}(A \lessdot B')=\mathscr{A}(A \lessdot B)$ if

either (i) $B'$ and $B$ are equi-width. i.e.,

$$m(B')\neq m(B) \quad \text{and} \quad w(B')=w(B)$$

or (ii) $B'$ and $B$ are equi-centred, i.e.,

$$m(B')=m(B) \quad \text{and} \quad w(B')\neq w(B).$$

This implies that through pairwise comparisons, the degree of acceptability (satisfaction) can be numerically tallied only in a set of intervals where one of the properties – either the central value or the width – remains unchanged.

**Remark 3.1.9.** If $B_1, B_2, \ldots, B_n$ be the equi-width intervals on $\mathfrak{R}$, such that

$$\mathscr{A}(B_1 \lessdot B_2)=\alpha_1\geqslant 0,$$
$$\mathscr{A}(B_2 \lessdot B_3)=\alpha_2\geqslant 0$$
$$\vdots$$

and

$$\mathscr{A}(B_{n-1} \otimes B_n) = \alpha_{n-1} \geqslant 0,$$

then

$$\mathscr{A}(B_1 \otimes B_n) = \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1}.$$

**Remark 3.1.10.** If $\mathscr{A}(B_1 \otimes B_2) = 0$ and $w(B_1) = w(B_2)$ then $B_1 \equiv B_2$, i.e. $B_1$ is identical to $B_2$.

Now, if $\mathscr{A}(B_1 \otimes B_2) = 0$ and $w(B_1) \neq w(B_2)$ then obviously $B_1 \not\equiv B_2$. But through $\mathscr{A}$-index, a direct comparison between them interpret that $B_1$ and $B_2$ are non-inferior to each other. Then the question arises: How to choose a preferred (maximizing) alternative?

**Proposition 3.1.11.** *Let us consider an interval $D = \langle m(D), w(D)\rangle$ which is inferior to the equi-centred intervals $B_1$ and $B_2$. Then, as compared to $D$, the grade of acceptability of superiority of the less uncertain interval is higher than that of superiority of the more uncertain interval. Symbolically, if*

$$\mathscr{A}(D \otimes B_1) > 0, \qquad \mathscr{A}(D \otimes B_2) > 0$$

*and*

$$\mathscr{A}(B_1 \otimes B_2) = 0, \text{ but } B_1 \not\equiv B_2,$$

*then,*

(i) $\mathscr{A}(D \otimes B_1) > \mathscr{A}(D \otimes B_2)$ *iff* $w(B_1) < w(B_2)$,
(ii) $\mathscr{A}(D \otimes B_1) < \mathscr{A}(D \otimes B_2)$ *iff* $w(B_1) > w(B_2)$.

**Proof.** The proof is straightforward.

Here condition (i) indicates that as compared to $D$, superiority of $B_1$ is more believable than the superiority of $B_2$. Hence, $B_1$ must be preferred to $B_2$ as maximizing alternative. This result is quite compatible to our intuition: if intervals $B_1$ and $B_2$ have the same expected value but $B_1$ contains less uncertainty than $B_2$ then $B_1$ is preferred to $B_2$.

Now, what happens if the reference interval $D^* = \langle m(D^*), w(D^*)\rangle$ is taken to be superior to the equi-centered interval $B_1$ and $B_2$, i.e., if

$$\mathscr{A}(B_1 \otimes D^*) > 0, \qquad \mathscr{A}(B_2 \otimes D^*) > 0$$

and

$$\mathscr{A}(B_1 \otimes B_2) = 0, \text{ but } B_1 \not\equiv B_2,$$

then,

(i) $\mathscr{A}(B_1 \otimes D^*) > \mathscr{A}(B_2 \otimes D^*)$ iff $w(B_1) < w(B_2)$,
(ii) $\mathscr{A}(B_1 \otimes D^*) < \mathscr{A}(B_2 \otimes D^*)$ iff $w(B_1) > w(B_2)$.

Here condition (i) states that as compared to $D^*$, inferiority of $B_1$ is more believable than the inferiority of $B_2$. Therefore, $B_1$ must be preferred to $B_2$ as better minimizing alternative. On the contrary, from this condition, $B_2$ cannot be said to be preferred to $B_1$ as maximizing alternative.

*Here is an important point to be noted*: The concept of acceptability index for comparing intervals in no way can be treated as analogous to the concept of 'difference' of real analysis. And for this reason, considering a superior reference interval $D^*$ for choosing a preferred maximizing alternative from among the equi-centred but not identical $B_1$ and $B_2$ or, considering an inferior reference $D$ for choosing a preferred minimizing alternative from among $B_1$ and $B_2$ do not make any sense and yield nothing.

### 3.2. Tong's Approach [11]

Tong deals with interval inequality relations in a separate way.

For a minimization problem as follows:

$$\text{Minimize} \quad Z = \sum_{j=1}^{n} [c_{\mathrm{L}j}, c_{\mathrm{R}j}] x_j,$$

$$\text{subject to} \quad \sum_{j=1}^{n} [a_{\mathrm{L}ij}, a_{\mathrm{R}ij}] x_j \geqslant [b_{\mathrm{L}i}, b_{\mathrm{R}i}] \quad \forall i = 1, 2, \ldots, m,$$

$$x_j \geqslant 0, \quad \forall j,$$

each inequality constraint is first transformed into $2^{n+1}$ crisp inequalities to yield

$$D_i = \{D_i^k / k = 1, 2, \ldots, 2^{n+1}\},$$

which are the solutions to the $i$th set of $2^{n+1}$ inequalities.

On the other hand, Tong defines a characteristic formula (CF)

$$\sum_{j=1}^{n} a_{ij} x_j \geqslant b_i$$

of the $i$th inequality relation, $\forall i$, where $a_{ij} \in [a_{\mathrm{L}ij}, a_{\mathrm{R}ij}]$ and $b_i \in [b_{\mathrm{L}i}, b_{\mathrm{R}i}]$.

Now, if the $i$th CF generates solution $D_i$ such that

$$D_i = \bigcup_{k=1}^{2^{n+1}} D_i^k,$$

then CF is called *maximum-value range inequality* and if CF generates solution $D_i$ such that

$$D_i = \bigcap_{k=1}^{2^{n+1}} D_i^k,$$

then it is called *minimum-value range inequality*. Tong [11] then defines minimum and maximum optimal objective value of the problem using max and min value inequalities, respectively.

### 3.3. Discussion

Let us take a very simple inequality relation with a single variable,

$[10, 20]x \leqslant [5, 35]$.

According to [11], the interval inequality generates $2^{1+1}$ crisp inequalities:

$$\left.\begin{array}{l} 10x \leqslant 5 \Rightarrow x \leqslant 0.5 \\ 10x \leqslant 35 \Rightarrow x \leqslant 3.5 \\ 20x \leqslant 5 \Rightarrow x \leqslant 0.25 \\ 20x \leqslant 35 \Rightarrow x \leqslant 1.75 \end{array}\right\} D = \{D^k/k = 1, 2, 3, 4\}$$

$$\overline{D} = \bigcup_{k=1}^{2^2} D^k \Rightarrow x \leqslant 3.5\text{: max value range inequality,}$$

$$\underline{D} = \bigcap_{k=1}^{2^2} D^k \Rightarrow x \leqslant 0.25\text{: min value range inequality,}$$

Here we would like to raise a question on Tong's approach [11]: how does one interpret the use of the operators *union* and *intersection* in defining max- and min-value range inequalities, respectively?

Using the union operator in defining the crisp equivalent form of the $i$th original constraint may be interpreted as *at least one element of the interval $A_i x$ is less than or equal to at least one element of interval $B_i$* which clearly does not validate the original constraint condition. Using $\mathscr{A}$-index it can be shown that $\mathscr{A}(B_i \oslash A_i x) = 1$, i.e., $A_i x$ is definitely greater than $B_i$.

On the other hand, using the intersection operator in defining the crisp equivalent form may be interpreted as *all elements of $A_i x$ is less than or equal to all elements of $B_i$* which is merely an oversimplification. Using $\mathscr{A}$-index, it can be shown that $\mathscr{A}(A_i x \oslash B_i) = 1$.

In actual practice, for a wide range of feasibility of the decision variable vector, DM may allow $A_i x$ even to be nested in $B_i$, i.e., some/all elements of $A_i x$ may even be allowed to be greater than or equal to some elements of $B_i$ and that how much to be allowed will be decided by the DM and this will depend on his optimistic attitude, on his risk versus benefit assessment and as a whole, on the level of satisfaction the DM tries to achieve from the decision-making process.

Hence, in our opinion, some sort of conditions indicating DM's satisfaction/utility requirement has to be incorporated in generation of a crisp equivalent structure of the inequality constraint with interval coefficients. Using the properties of *$\mathscr{A}$-index* we develop a satisfactory crisp equivalent structure of an inequality constraint with interval coefficients.

### 3.4. A satisfactory crisp equivalent system of $Ax \leqslant B$

Let $A = [a_L, a_R], B = [b_L, b_R]$ and $x$ is a singleton variable.

According to $\mathscr{A}$-index the acceptability condition of $Ax \leqslant B$ may be defined as

$\mathscr{A}(Ax \oslash B) \geqslant 0$,

i.e. $m(Ax) \leqslant m(B)$.

Now, let us take the condition $m(Ax) = m(B)$, then, for a given value of $x$, we may have two different possible setups.

*Case*-I: When interval $A$ is relatively narrower than interval $B$: $Ax$ may be nested in $B$. For example, for $x = 2$, the relation $[2, 4]x \leqslant [2, 10]$ may be viewed as given in Fig. 1.

*Case*-II: When interval $A$ is relatively wider than it was in case I: $B$ may be nested in $Ax$. For example, for $x = 2$, the relation $[0, 6]x \leqslant [2, 10]$ may be viewed as shown in Fig. 2.

From the examples given above, for both of the cases, the following remarks may be made:

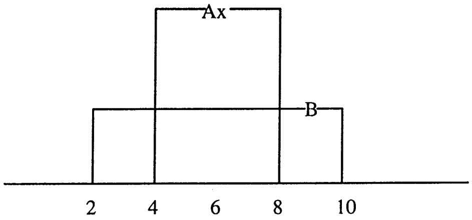


Fig. 1.

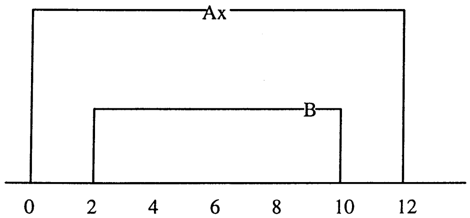


Fig. 2.

(i) Case I definitely satisfies the original interval inequality for $x \leqslant 2$ because $\mathscr{A}(Ax \otimes B) \geqslant 0$. However, an optimistic DM may remain under-satisfied with the optimal constraint condition and for getting higher satisfaction, he may like to increase the value of $x$ to such an extent that $\mathscr{A}(B \otimes Ax)$ does not pass over a threshold assumed and fixed by him.

(ii) On the other hand, by case II, the original interval inequality condition is not denied even for $x \leqslant 2$ because $\mathscr{A}(Ax \otimes B) \geqslant 0$. But a pessimistic DM may not be satisfied if the right limit of $Ax$ spills over the right limit of $B$. To attain his required level of satisfaction the DM may even like to reduce the value of $x$ so that $a_R x \leqslant b_R$.

To shed more light on interval inequality relation from a different angle, let us refer an equivalent form of a deterministic inequality where

$ax \leqslant b$ is $ax \in ]-\infty, b]$.

Let us extend this concept to an inexact environment: if the real numbers $a$ and $b$ are allowed to be replaced by intervals $A$ and $B$, respectively, that one's possible reaction is as much as similar to Moore's concept of set-inclusion, i.e.,

$$Ax \leqslant B \;\Rightarrow\; Ax \subset D$$

where $D = ]-\infty, b_R]$.

Keeping in view the two remarks stated above and the Moore's concept [7], we propose a satisfactory crisp equivalent form of interval inequality relation as follows:

$$Ax \leqslant B \;\Rightarrow\; \begin{cases} a_R x \leqslant b_R, \\ \mathscr{A}(B \otimes Ax) \leqslant \alpha \in [0, 1], \end{cases}$$

where, $\alpha$ may be interpreted as an optimistic threshold assumed and fixed by the DM.

Similarly, for $Ax \geqslant B$, we have the satisfactory crisp equivalent form by the following pair:

$$a_L x \geqslant b_L,$$

$$\mathscr{A}(Ax \otimes B) \leqslant \alpha \in [0,\ 1].$$

## 4. An interval linear programming problem and its solution

Let us consider the following problem:

$$\text{Minimize} \quad Z = \sum_{j=1}^{n} [c_{Lj}, c_{Rj}] x_j$$

$$\text{subject to} \quad \sum_{j=1}^{n} [a_{Lij}, a_{Rij}] x_j \geqslant [b_{Li}, b_{Ri}],$$

$$\forall i = 1,\ 2, \ldots, m,$$

$$x_j \geqslant 0, \quad \forall j.$$

As is described in the previous section a satisfactory crisp equivalent system of constraints of the $i$th interval constraint can be generated as follows:

$$\sum_{j=1}^{n} a_{Lij} x_j \geqslant b_{Li}, \quad \forall i,$$

$$b_{Li} + b_{Ri} - \sum_{j=1}^{n} (a_{Lij} + a_{Rij}) x_j \leqslant \alpha(b_{Ri} - b_{Li}) + \alpha \sum_{j=1}^{n} (a_{Rij} - a_{Lij}) x_j.$$

The working of $\mathscr{A}$*-index* may be summarized by the following principle:

The position (of mean) of an interval compared to that of another reference interval results in whether the former is superior or inferior to the later. On the other hand, the width of a superior (inferior) interval compared to that of the reference interval specifies the grade to which the DM is satisfied with the superiority (inferiority) of the former compared to the later.

The objective of a conventional linear programming problem is to maximize or minimize the value of its (one only, single-valued) objective function satisfying a given set of restrictions. But, a single-objective interval linear programming problem contains an interval-valued objective function. As an interval can be represented by any two of its four attributes (viz., left limit, right limit, mid-value and width) [4], an interval linear programming, by using attributes mid-value and width (say) can be reduced into a linear biobjective programming problem as follows:

Max/Min {mid-value of the interval objective function},

Min {width of the interval objective function},

sub. to {set of feasibility constraints}.

From this problem naturally one may get two conflicting optimal solutions:

$$x^* = \{x_j^*\} \text{ from } \max/\min \{\text{mid value}\} \text{ sub. to } \{\text{constraints}\},$$

$$x^{**} = \{x_j^{**}\} \text{ from } \min \{\text{width}\} \text{ sub. to } \{\text{constraints}\}$$

and from there two optimal interval values $Z^*$ and $Z^{**}$.

If $x^* = x^{**}$, then there does not exist any conflict and $x^*$ is the solution of the problem:

But if $x^* \neq x^{**}$, for the maximization problem,

$$m(Z^*) > m(Z^{**}) \quad \text{and} \quad w(Z^*) > w(Z^{**})$$

(because, $Z^*$ is obtained through maximizing $m(Z)$ and $Z^{**}$ is obtained not by maximizing $m(Z)$, but through another goal, by minimizing $w(Z)$).

Similarly, for minimizing problem, if $x^* \neq x^{**}$, then,

$$m(Z^*) < m(Z^{**}) \quad \text{and} \quad w(Z^*) > w(Z^{**})$$

(because $Z^*$ here is obtained by minimizing $m(Z)$ and $Z^{**}$ by minimizing $w(Z)$).

Therefore, if $x^* \neq x^{**}$, $Z^*$ and $Z^{**}$ are said to be two non-dominated alternative extreme interval objective values [3].

On the other hand, the principle of $\mathscr{A}$-index indicates that for the maximization (minimization) problem, an interval with a higher mid-value is superior (inferior) to an interval with a lower mid-value. Therefore, though $Z^*$ and $Z^{**}$ are two non-dominated alternatives from the viewpoint of a biobjective problem, as two interval values of the interval-valued objective function of the original problem they can be ranked. Hence, in order to obtain max/min of the interval objective function, considering the mid-value of the interval-valued objective function is our primary concern. We reduce the interval objective function its central value and use conventional LP techniques for favour of its solution. We also consider width but as a secondary attribute, only to confirm whether it is within the acceptable limit of the DM. If it is not, one has to reduce the extent of width (uncertainty) according to his satisfaction and thus to obtain a less wide interval from among the non-dominated alternatives accordingly.

The following LP problem is the necessary equivalent form of the original problem:

$$\begin{aligned}
\text{Minimize} \quad & m(Z) = \frac{1}{2}\sum_{j=1}^{n}(c_{\mathrm{L}j} + c_{\mathrm{R}j})x_j \\
\text{subject to} \quad & \sum_{j=1}^{n} a_{\mathrm{L}ij}x_j \geqslant b_{\mathrm{L}i}, \quad \forall i, \\
& b_{\mathrm{L}i} + b_{\mathrm{R}i} - \sum_{j=1}^{n}(a_{\mathrm{L}ij} + a_{\mathrm{R}ij})x_j \\
& \quad \leqslant \alpha(b_{\mathrm{R}i} - b_{\mathrm{L}i}) + \alpha\sum_{j=1}^{n}(a_{\mathrm{R}ij} - a_{\mathrm{L}ij})x_j, \\
& x_j \geqslant 0, \quad \forall j.
\end{aligned}$$

It is only when there exists the possibility of multiple solution, that comparative widths are required to be calculated and then in favour of a minimum available width, we get the solution.

### 4.1. Solution to the problem stated in Section 1

$$\begin{aligned}
\text{Minimize} \quad & 0.4x_1 + 0.2x_2 \\
\text{subject to} \quad & x_1 + x_2 \geqslant 1000, \\
& x_1 + x_2 \leqslant 1130, \\
& 0.48x_1 + 0.85x_2 \geqslant 210, \\
& 0.51x_1 + 1.0075x_2 \geqslant 215, \\
& 5x_1 + 3x_2 \geqslant 4000, \\
& 8x_1 + 3x_2 \geqslant 4500, \\
& x_1, x_2 \geqslant 0.
\end{aligned}$$

(The DM here assumes that $\alpha = 0.5$.)

**Result.** $x_1 = 305$, $x_2 = 825$ and $Z_{0.5} = [280.9, 293.1] = \langle 287, 6.1 \rangle$ (where, $Z_{0.5}$ is the minimum objective value at $\alpha = 0.5$ and $\langle 287, 6.1 \rangle$ is the alternative representation of $[280.9, 293.1]$, indicating $\langle$mid value, width$\rangle$).

Tong's approach gives the following solution from the same problem:

$$x_1 = [x_1', x_1''] = [234.57, 1050],$$

$$x_2 = [x_2', x_2''] = [765.43, 250]$$

and

$$Z_{\text{Tong}} = [242.22, 491] = \langle 366.61, 124.39 \rangle.$$

As far as the problem's minimizing objective is concerned, our solution gives better expected value with far better certainty. Also by using $\mathscr{A}$-index, we have, $\mathscr{A}(Z_{0.5} \otimes Z_{\text{Tong}}) = 0.610$.

Clearly, this indicates that the solution with our approach is much better than that with Tong's approach.

Even if we compute the satisfying conditions of the crisp equivalent set of constraints at $\alpha = 0$, we get the solution to the above problem as

$$x_1 = 571.28, \quad x_2 = 428.57$$

and

$$Z_0 = [302.85, 325.71] = \langle 314.28, 11.42 \rangle$$

for which also $\mathscr{A}(Z_0 \otimes Z_{\text{Tong}}) = 0.385$. This indicates that even at $\alpha = 0$ (at a no optimism level) our solution is better than Tong's solution.

## 5. Conclusion

The aim of this paper was to define a satisfactory crisp equivalent system of an inequality constraint with interval coefficients. The approach defined here has come out as an application of $\mathscr{A}$*-index* for comparing two intervals through DM's satisfaction. Once the crisp equivalent structure of the constraint set is defined, solution to a problem with maximizing or minimizing objective function practically turns to be maximization or minimization of the central value of the interval-valued objective function. In this regard, a point worth mentioning is: If the DM is not satisfied with the extent of uncertainty (width) involved in the optimal objective value, he can achieve his required level of satisfaction by adjusting allowable width of the optimal objective value and/or by redefining satisfying conditions for generating crisp equivalent set of constraints. An interactive approach through which a DM learns gradually more and more about the problem and where his feedback is used to guide the solution of the problem to a more favourable one, may be formulated as a generalized and more credible procedure for solving interval linear programming problems.

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