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# Rationally triangulable automorphisms

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*Abstract*

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This paper provides a necessary and sufficient condition for the rational triangulability of actions of the algebraic group $G_a$ on affine space. The criterion is used to demonstrate the rational triangulability of all $G_a$ actions on $A^3(k)$, as well as to prove, for arbitrary $n$, that all $G_a$ actions are stably rationally triangulable.

## 1. Introduction

A rational action of an algebraic group $G$, defined over the characteristic zero, algebraically-closed field $k$, on the affine space $A^n(k)$, is said to be *triangulable* if coordinates $x_1, \ldots, x_n$ can be chosen so that the induced automorphism on the coordinate ring has the form $x_i \mapsto \alpha_i x_i + F_i(x_1, \ldots, x_{i-1})$ with $\alpha_i$ in the multiplicative group of $k$. The action is said to be *linear* if there is a coordinate system on which it is effected by a linear change of variables, and *tame* if it lies in the group generated by the triangular and linear automorphisms.

It is known that the automorphism group of $A^2(k)$ is the amalgamated free product of the groups of linear and triangular automorphisms, but it remains unknown whether these subgroups generate the automorphism group if $n \geq 3$. Bass, in [1], and Popov, in [4], have given examples of actions of the additive group of $k$, denoted $G_a$, on $A^3(k)$ which are neither linearizable nor triangulable. The structure theory of amalgamated products thus shows that the automorphism group cannot have this structure for $n \geq 3$.

Two approximations to tameness are the notions of stable tameness and rational triangulability. An action of $G$ on $A^n(k)$ is *stably tame* provided its extension to $A^{n+m}(k)$ by fixing the last $m$ coordinates is tame, and *rationally* *triangulable* if there are generators $y_1, \ldots, y_n$ of the field of rational functions so that each of the subfields $k(y_1, \ldots, y_i)$ is invariant under the group of $k$-automorphisms of the rational function field induced by $G$. In [6], Smith showed that the examples of Popov are stably tame. It was asked in [1] whether every rational action of a unipotent group on affine space is rationally triangulable.

This paper provides a necessary and sufficient condition for the rational triangulability of actions of the additive group of $k$ on affine space. The criterion can be used to demonstrate the rational triangulability of all $G_a$ actions on $A^3(k)$, in particular those of [1] and [4], as well as to prove, for arbitrary $n$, that all $G_a$ actions are stably rationally triangulable (indeed they are rationally triangulable in the extension of the action to $A_{n+1}(k)$).

## 2. Generation of purely transcendental extensions

We begin with a general result on pure transcendental extensions of degree one of an arbitrary field of characteristic zero.

**Theorem 2.1.** *Let $K$ be a field of characteristic zero, not necessarily algebraically closed, and $K(z)$ a simple transcendental extension. An element $w \in K(z)$ satisfies $K(w) = K(z)$ if and only if there is an automorphism $f$ of $K(z)$ fixing $K$ and sending $w$ to $w + c$ for some nonzero $c \in K$.*

**Proof.** If $K(z) = K(w)$, then $f(w) = w + 1$ is the desired automorphism. Conversely suppose that $f$ is a $K$-automorphism of $K(z)$ mapping $w$ to $w + c$ for some nonzero $c$ in $K$. Then the group $\langle f \rangle$ of $K$-automorphisms generated by $f$ is infinite, translates $w$, and leaves $K(w)$ invariant.

Since $f$ fixes $K$ and moves $w$, $K(z)$ is a finite-dimensional, separable extension of $K(w)$; in particular, only finitely many places of $K(w)$ over $K$ ramify in $K(z)$. Since $K(w)$ is $\langle f \rangle$-invariant, those places are permuted by the $\langle f \rangle$ action and therefore an infinite subgroup $H$ of $\langle f \rangle$ fixes them.

Let $h \in H$ with $h(w) = w + a$, for some $0 \neq a \in K$, and let $\mathcal{P}$ be a ramified place. Then $h\mathcal{P} = \mathcal{P}$ and $\mathcal{P}(w) = (h\mathcal{P})(w) = \mathcal{P}(w + a) = \mathcal{P}(w) + a$. Since $a \neq 0$, $\mathcal{P}(w) = \infty$. Thus the only places which ramify are the poles of $w$.

Let $[K(z) : K(w)] = n$ and let $\bar{K}$ be an algebraic closure of $K$. Then $[\bar{K}(z) : \bar{K}(w)] = n$ and again ramification can occur only at the places of $\bar{K}(w)$ which are poles of $w$. The remainder of the proof follows as in [8, p. 232]. Namely, with $G$ (resp. $g$) denoting the genus of $\bar{K}(z)$ (resp. $\bar{K}(w)$), $\mathcal{D}$ the different, and $d^0(\mathcal{D})$ its degree, the Hurwitz–Zeuthen formula $2G - 2 - n(2g - 2) = d^0(\mathcal{D})$ yields $d^0(\mathcal{D}) = 2n - 2$, since $G = g = 0$. But the concentration of the ramification at the poles of $w$ implies that $d^0(\mathcal{D}) \leq n - 1$. Thus, $n = 1$. $\square$

The theorem clearly does not hold in positive characteristic $p$. One simply takes $w = z^p$ and $g(z) = z + 1$.

The following result shows that rationally triangulable actions of $G_a$ have a particularly simple form.

**Theorem 2.2.** *If $G = G_a$, acting rationally and nontrivially on $A^n(k)$, is rationally triangulable, then $k(x_1, \ldots, x_n) = k(z_1, \ldots, z_n)$ where $k(z_1, \ldots, z_{n-1})$ is fixed by $G$, and for all $\sigma \in G$, $\sigma(z_n) = z_n + t_\sigma$ for $t_\sigma \in k(z_1, \ldots, z_{n-1})$.*

**Proof.** Let $y_1, \ldots, y_n$ generate $k(x_1, \ldots, x_n)$ with the fields $k(y_1, \ldots, y_i)$ invariant under the given $G_a$ action. By Rosenlicht's cross-section theorem, $k(y_1, y_2) = K^G(w)$ where $K^G$ is the fixed field of $G$ in its restriction to $k(y_1, y_2)$, and $w$ is transcendental over $k$ [2, p. 152]. By the generalized Luroth theorem [3], $K^G = k(z_1)$. Thus $k(x_1, \ldots, x_n) = k(z_1)(w_1, y_3, \ldots, y_n)$. By induction, it follows that $k(y_1, \ldots, y_n) = k(z_1, \ldots, z_{n-1})(w_n)$ and that $k(z_1, \ldots, z_{n-1})$ is fixed by $G$.

Since $G$ acts rationally and nontrivially on $k[x_1, \ldots, x_n]$, there is a finite-dimensional generating subspace $V$ on which the action can be represented by unipotent matrices, and an element $w \in V$ for which $\sigma(w) = w + t_\sigma$ for all $\sigma \in G$. It follows from Theorem 2.1 that $k(z_1, \ldots, z_{n-1})(w_n) = k(z_1, \ldots, z_n)(w)$. □

## 3. Rationally and stably rationally triangular actions

A rational action of an algebraic group $G$ on an affine domain $A$ over $k$ has a unique extension to an action on the field of fractions $K$. The subfield of $K$ fixed elementwise by the extended action will be denoted $K^G$.

**Theorem 3.1.** *Every rational action of $G_a$ on $k[x_1, \ldots, x_n]$ is stably rationally triangulable. An action is rationally triangulable if and only if $k(x_1, \ldots, x_n)^{G_a}$ is a pure transcendental extension of $k$.*

**Proof.** Let $k^{(n)} = k(x_1, \ldots, x_n)$ and $F = k^{(n)G_a}$. Then $k^{(n)} = F(w)$ and $G_a$ acts as translations on $w$. The second assertion is therefore obvious. However, $F(w)$ is a pure transcendental extension of $k$, and so therefore is $F(x_{n+1})$ for a new variable $x_{n+1}$. The action of $G_a$, extended to $k^{(n)}(x_{n+1})$ by fixing this variable, is therefore rationally triangulable. □

**Corollary 3.2.** *Every rational $G_a$ action on $k[x_1, x_2, x_3]$ is rationally triangulable.*

**Proof.** According to Castlenuovo's theorem, a unirational field of transcendence degree 2 over an algebraically-closed field of characteristic zero is pure transcendental. This applied to $F$ of the previous theorem yields the result. □

A finite-dimensional linear representation of $G_a$ in $\mathrm{GL}(V)$ induces an action on the affine space spec $S(V)$, where $S(V)$ is the symmetric algebra of $V$. Such an action is called a *linear* $G_a$ *action*. Alternatively, a nilpotent endomorphism of $V$ extends to a locally nilpotent derivation of $S(V)$, which can be exponentiated to yield a one-parameter group of automorphisms of $S(V)$ isomorphic to $G_a$. Indeed, all linear $G_a$ actions arise in this way. If $\delta$ is such a derivation and $f \in S(V)$ one of its constants, then $t \mapsto \exp(tf\delta)$ is a $G_a$ action since $f\delta$ is at least locally nilpotent on $S(V)$. The examples of Bass [1] and Popov [4] are precisely of this form, and so will be called *Popov* $G_a$ actions.

**Corollary 3.3.** *All Popov* $G_a$ *actions are rationally triangulable.*

**Proof.** The Jordan normal form of a nilpotent endomorphism of $V$ shows that a linear $G_a$ action is triangulable (a fortiori rationally so). As such the fixed field is pure transcendental over $k$. However, the fixed field of a Popov action is identical to that of the linear action from which it was derived. □

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