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Computational aspect to the nearest matrix with two prescribed eigenvalues

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ABSTRACT

Given a complex square matrix A and two complex numbers λ_1 and λ_2, we present a method to calculate the distance from A to the set of matrices X that have λ_1 and λ_2 as some of their eigenvalues. We also find the nearest matrix X.

1. Introduction

In the recent paper [1], Malyshev obtained the following formula for the 2-norm distance $\mathrm{rsep}(A)$ from a complex $n \times n$ matrix to a closest matrix with a multiple eigenvalue:

$$\mathrm{rsep}(A) = \min_{\lambda \in \mathbf{C}} \max_{\gamma \geq 0} \sigma_{2n-1}(G(\gamma)). \tag{1}$$

Here,

$$G(\gamma) = \begin{pmatrix} \lambda I - A & \gamma I \\ 0 & \lambda I - A \end{pmatrix},$$

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and $\sigma_{2n-1}(G(\gamma))$ is the penultimate singular value of the matrix G, assuming that the singular values are numbered in decreasing order. Ikramov and Nazari in [3] introduced a correction for Malyshev's formula when A be a normal matrix. Lippert in [4] has presented a generalization for Malyshev's problem, finding $\|\Delta M\|_2$ optimal perturbations of M such that $M - \Delta M$ has two given eigenvalues. In 2005, Gracia [2] extended formula (1) for two prescribed eigenvalues, in the following theorems. Define

$$F(\gamma) = \begin{pmatrix} \lambda_1 I - A & \gamma I \\ 0 & \lambda_2 I - A \end{pmatrix}.$$

Theorem 1. *Let $\gamma^* > 0$ be a local optimizer of function $f(\gamma) = \sigma_{2n-1}(F(\gamma))$. Suppose*

$$\sigma^* = f(\gamma^*) > 0,$$

then there exists a pair of normalized singular vectors associated with the singular value σ^* of $F(\gamma^*)$, namely a left vector

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad u_1, u_2 \in \mathbf{C}^n$$

and a corresponding right vector

$$v = \frac{1}{\sigma^*} F(\gamma^*)^* u = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad v_1, v_2 \in \mathbf{C}^n$$

such that

$$\mathrm{Re}(u_1^* v_2) = 0. \tag{2}$$

Moreover, the matrices

$$U = (u_1 \quad u_2), \quad V = (v_1 \quad v_2) \tag{3}$$

satisfy the relation

$$U^* U = V^* V. \tag{4}$$

Theorem 2. *If γ^* in Theorem 1 is a positive number, then both matrices in (3) have rank 2. The matrix $B = A + \Delta$, where*

$$\Delta = \sigma^* U V^{\dagger}, \tag{5}$$

is the closest (with respect to the 2-norm) matrix to A having eigenvalues λ_1 and λ_2 and

$$\|\Delta\|_2 = \sigma^*. \tag{6}$$

Below, we discuss some issues related to the computer implementation of this method. It turns out that the case of a general matrix A is substantially different from that of a normal matrix A.

2. Normal matrix

Let A be a normal matrix. We illustrate by a specific example. If

$$A = \begin{pmatrix} 7 & 0 & 0 & 0 & 1 \\ 0 & 8 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 1 & 2 & 0 & 1 & 7 \end{pmatrix},$$

and $\lambda_1 = 0, \lambda_2 = 2$, we found the values

$$\gamma^* = 1.05800, \quad \sigma^* = 1.84331.$$

The singular value $\sigma_{2n-2}(F(\gamma^*))$ equals 1.84382. These two values are approximately the same, namely

$$\sigma_{2n-1}(F(\gamma^*)) \simeq \sigma_{2n-2}(F(\gamma^*)).$$

Thus, in the optimal matrix $F(\gamma^*)$, the value σ^* is iterated.

Let $u^{(2n-1)}, v^{(2n-1)}$ and $u^{(2n-2)}, v^{(2n-2)}$ be the pairs of singular vectors of $F(\gamma^*)$ associated with σ_{2n-1} and σ_{2n-2}, respectively, that MATLAB gives us. An attempt to use any of these pairs for implementing the construction described in Theorem 2 leads to catastrophic results. Namely, for the matrix $\Delta^{(2n-1)} = -\sigma^* U^{(2n-1)} V^{(2n-1)\dagger}$, we obtain

$$\|\Delta^{(2n-1)}\| = 1.49826 \times 10^{12},$$

while $\Delta^{(2n-2)} = -\sigma^* U^{(2n-2)} V^{(2n-2)\dagger}$ has the norm

$$\|\Delta^{(2n-2)}\| = 6.26340 \times 10^{12}.$$

It is easy to find the reason why equality (6) is violated in both cases. Calculating $u_1^* v_2$, we obtain for the pair $u^{(2n-1)}, v^{(2n-1)}$

$$-0.14488$$

and for the pair $u^{(2n-2)}, v^{(2n-2)}$

$$0.60502.$$

In any case above, equality (2), even approximately does not hold. It follows that equality (4) is violated.

The situation can be rectified as follows. Consider the number

$$\sigma^* = \sigma_{2n-1}(F(\gamma^*))$$

as a double singular value of $F(\gamma^*)$ and the vectors $u^{(2n-1)}$ and $u^{(2n-2)}$ as an orthonormal basis in the left singular subspace associated with σ^*. In this subspace, we look for a normalized vector

$$u = \alpha u^{(2n-1)} + \beta u^{(2n-2)}, \quad |\alpha|^2 + |\beta|^2 = 1, \tag{7}$$

and combined with the associated right singular vector

$$v = \alpha v^{(2n-1)} + \beta v^{(2n-2)} \tag{8}$$

in order to satisfy relation (2).

From (2) we have

$$\mathrm{Re}(u_1^* v_2) = 0. \tag{9}$$

Substituting (7) and (8) into (9), we achieve the relation

$$(\bar{\alpha} \quad \bar{\beta})\mathrm{Re}W \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0, \tag{10}$$

in which

$$W = \begin{pmatrix} u_1^{(2n-1)*} v_2^{(2n-1)} & u_1^{(2n-1)*} v_2^{(2n-2)} \\ u_1^{(2n-2)*} v_2^{(2n-1)} & u_1^{(2n-2)*} v_2^{(2n-2)} \end{pmatrix}, \tag{11}$$

and

$$W_r = \mathrm{Re}W = \begin{pmatrix} \mathrm{Re}W_{11} & (W_{12} + \overline{W_{21}})/2 \\ (\overline{W_{12}} + W_{21})/2 & \mathrm{Re}W_{22} \end{pmatrix}. \tag{12}$$

The existence of a nontrivial solution for Eq. (10) is ensured by the fact that the Hermitian matrix (10) is indefinite. In fact, let us call $g(\gamma) = \sigma_{2n-2}(F(\gamma))$. Let $\mu_1 \geqslant \mu_2$ be the eigenvalues of the matrix ReW. Then the right derivatives of the functions f and g at γ^* are equal to μ_2 and μ_1

$$f'(\gamma^{*+}) = \mu_2, \quad g'(\gamma^{*+}) = \mu_1,$$

respectively. Since f is decreasing and g is increasing at right of γ^*, we deduce that

$$\mu_2 < 0 \quad \text{and} \quad \mu_1 > 0.$$

The numbers α and β can be found, for example, in the following manner. Let

$$W_r = PMP^*, \quad M = \mathrm{diag}(\mu_1, \mu_2),$$

be the spectral decomposition of W. Set

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = P\begin{pmatrix}\gamma\\ \delta\end{pmatrix}, \tag{13}$$

and recast (10) as

$$\mu_1|\gamma|^2 + \mu_2|\delta|^2 = 0, \quad |\gamma|^2 + |\delta|^2 = 1. \tag{14}$$

The pair

$$\left(\frac{|\mu_2|}{|\mu_1| + |\mu_2|}\right)^{\frac{1}{2}}, \quad \left(\frac{|\mu_1|}{|\mu_1| + |\mu_2|}\right)^{\frac{1}{2}}$$

is a solution to system (14). (Recall again that μ_1 and μ_2 are numbers of different signs.) Using (13), we obtain the corresponding pair α, β.

In the example above with matrix A, this technique yields

$$\alpha = -0.89822, \quad \beta = -0.43955.$$

For the corresponding singular vectors (7) and (8), we have

$$u_1^* v_2 = -1.45717 \times 10^{-16}.$$

The matrix Δ constructed from these vectors has the norm

$$1.84350,$$

which is in very good agreement with σ^*. Finally, we found

$$U^*U = \begin{pmatrix}0.09189 & 0.21587\\ 0.21587 & 0.90811\end{pmatrix}, \quad V^*V = \begin{pmatrix}0.09191 & 0.21590\\ 0.21590 & 0.90809\end{pmatrix},$$

it follows that $U^*U \simeq V^*V$.

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