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On an optimal finite element scheme for the advection equation

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ABSTRACT

In this report it is presented a numerical finite element scheme for the advection equation that attains the optimal L^2 convergence rate $\mathcal{O}(h^{k+1})$ when order k finite elements are used, improving the order $\mathcal{O}(h^{k+0.5})$ of other previous regularization methods. This result is also confirmed by the numerical test performed in the last section. The scheme assumes unstructured grids, periodic boundary conditions, a constant advection field and a bit (two units) stronger regularity on the exact solution than in the classical (suboptimal) finite element theory.

1. Introduction

In general the $L^\infty(L^2)$-error of the standard finite element Galerkin method for first order hyperbolic systems converges in the order $\mathcal{O}(h^k)$ where h is the mesh size and k the order of the finite elements, see Dupont, [1], one unit less than expected. Optimal convergence has been proved only in some particular cases (such as linear elements or cubic splines on uniform grids and for periodic boundary conditions, see Dupont [1], Thomee and B. Wendroff, [2]).

Various regularization methods have been employed to improve the convergence rate on unstructured grids. In the class of filter based regularization methods, such as the one used here we mention Layton and Connors, [3], Ervin and Jenkins, [4], Dunca and Neda [5]. If periodic boundary conditions are assumed and unstructured grids and order k elements are used, to the author's knowledge the best convergence rate available in the literature is $\mathcal{O}(h^{k+0.5})$, see for example the models in Layton and Connors, [3], or Dunca and Neda [5].

This paper considers a numerical scheme to solve the model advection equation

$$u_t + \vec{a} \cdot \nabla u = f, \quad u(0) = u_0 \tag{1}$$

optimally in case $\vec{a}$ is a constant vector and periodic boundary conditions are assumed. The mesh and the finite element spaces X_h are chosen such that a general approximation property, see inequality (4), holds. The exact solution is not necessarily smooth, but it should be a bit more regular (two powers) than in the classical suboptimal theory.

The algorithm presented herein is based on the idea developed in the papers of Dunca, John and Layton, [6,7], which is that, in some cases, the mean finite element error has a higher convergence rate than the finite element error itself. Here the mean $\overline{v}$ of v is computed using the differential filter (on the fixed length scale $\delta = 1$), see Germano, [8], Dunca and John, [6],

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$\overline{v} = \mathcal{S}(v)$, where

$$\overline{v} - \Delta\overline{v} = v. \tag{2}$$

To approximate optimally the solution u of Eq. (1), we first apply the differential operator $I - \Delta$ to Eq. (1) to get

$$(I-\Delta)u_t + \vec{a}\cdot\nabla(I-\Delta)u = (I-\Delta)f, \qquad (I-\Delta)u(0) = (I-\Delta)u_0.$$

Therefore $(I-\Delta)u$ is the solution w of the problem

$$w_t + \vec{a}\cdot\nabla w = (I-\Delta)f, \qquad w(0) = (I-\Delta)u_0 \tag{3}$$

i.e., $w = (I-\Delta)u$, and therefore, using Eq. (2), we obtain $u = \overline{w}$.

In this regard we may view the solution u of problem (1) as being the exact average $\overline{w}$ of the solution w of problem (3). As such, one expects better convergence rate if, instead of solving directly (and suboptimally) with FEM problem (1), one first solves with FEM problem (3) to get w_h (which is a suboptimal approximation of w) and then filters w_h to obtain $\overline{w_h}^h$. In Section 4 we prove that, if u satisfies several regularity assumptions, then $\overline{w_h}^h$ is indeed an optimal approximation of $\overline{w} = u$, i.e. $\|\overline{w_h}^h - u\|_{L^\infty([0,T],L^2(\Omega))}$ is $\mathcal{O}(h^{k+1})$ where k is the order of the finite elements.

2. Mathematical setting

We let Ω be the 2d or 3d periodic box. The norm $\|\cdot\|$ will denote the usual L^2 norm on Ω and $(\cdot,\cdot)$ will be the corresponding L^2 inner product on Ω. For a given natural number k, H^k will denote the usual Sobolev space of order k on Ω and $\|\cdot\|_k$ and $|\cdot|_k$ are its usual Sobolev norm and seminorm respectively.

$H^k_\#(\Omega)$ will denote the closure of the smooth, periodic functions defined on Ω in the Sobolev $\|\cdot\|_k$ norm. For $k=1$ we let $X = H^1_\#(\Omega)$ and for $k=0$ we let $L^2_\#(\Omega) = H^0_\#(\Omega)$.

In the sequel $X_h \subset X$ will denote a conforming finite element space on a quasi-uniform mesh of size h on Ω satisfying the general approximation assumption that there exists a general constant C such that

$$\|v - v_h\| + h\|\nabla v - \nabla v_h\| \leq Ch^{l+1}|v|_{l+1} \tag{4}$$

for some interpolant $v_h \in X_h$ of $v \in X \cap H^{l+1}$, where $1 \leq l \leq k$.

For $u \in L^2_\#(\Omega)$ its mean $\overline{u} \in H^2_\#(\Omega) \subset L^2_\#(\Omega)$ is defined using the differential filter, see Germano, [8], as the unique solution of the PDE

$$-\Delta\overline{u} + \overline{u} = u \tag{5}$$

with periodic boundary conditions. We let $\mathcal{S} : L^2_\#(\Omega) \to L^2_\#(\Omega)$, $\mathcal{S}u = \overline{u}$, denote the differential filtering operator.

We also let the discrete mean $\overline{u}^h \in X_h$ of u to be the classical FEM approximation of $\overline{u}$, defined by

$$(\nabla\overline{u}^h, \nabla v_h) + (\overline{u}^h, v_h) = (u, v_h)$$

for any $v_h \in X_h$. We let $\mathcal{S}_h : L^2_\#(\Omega) \to L^2_\#(\Omega)$, $\mathcal{S}_h u = \overline{u}^h$, denote the discrete differential filtering operator.

Remark 2.1. One can show, see [6,9,10] that the differential filtering operators $\mathcal{S}$, $\mathcal{S}_h$ are selfadjoint and they satisfy the stability inequality

$$\|\overline{v}\| \leq \|v\|, \qquad \|\overline{v}^h\| \leq \|v\|, \quad \forall v \in L^2_\#(\Omega). \tag{6}$$

The following known result states the classical FEM convergence rate of the elliptic second order PDE (5), obtained using Céa's lemma and the Aubin–Nitsche duality method, see Brenner and Scott, Theorem 5.7.6 on page 144, [11].

Remark 2.2. For $u \in X$ there holds

$$\|\overline{u} - \overline{u}^h\| + h\|\nabla\overline{u} - \nabla\overline{u}^h\| \leq Ch^2\|\overline{u}\|_2 \leq Ch^2\|u\|. \tag{7}$$

In case $u \in H^{k-1}_\#(\Omega)$ we have that

$$\|\overline{u} - \overline{u}^h\| + h\|\nabla\overline{u} - \nabla\overline{u}^h\| \leq Ch^{k+1}\|\overline{u}\|_{k+1} \leq Ch^{k+1}\|u\|_{k-1}. \tag{8}$$

Here C is a general constant not depending on u or h.

3. Estimates of the mean and discrete mean errors of the classical FEM

We assume that $u \in C^1([0,T],X)$ is the solution of the model advection equation in variational formulation:

$$\left(\frac{du}{dt}, v\right) + (\vec{a}\cdot\nabla u, v) = (f, v), \quad \text{for all } v \in X, \tag{9}$$

and $u(0) = u_0$ where $u_0 \in X, f \in L^2([0,T],X)$ and $\vec{a}$ is a constant vector.

We will further assume that $u, u_t \in L^2([0,T], H_{\#}^{k+1}(\Omega))$.

We will let $u_h \in C^1([0,T],X_h)$ be the semidiscrete finite element approximation of u, i.e. u_h satisfies

$$\left(\frac{du_h}{dt}, v_h\right) + (\vec{a}\cdot\nabla u_h, v_h) = (f, v_h) \quad \text{for all } v_h \in X_h, \tag{10}$$

and $u_h(0)$ is the interpolant of $u(0)$ that satisfies the approximation assumption in inequality (4).

We will let e denote the error $e(t) = u(t) - u_h(t) \in X$ for $t \in [0,T]$.

Classical theory shows that there holds the suboptimal estimate

$$\|e\|_{L^\infty([0,T],L^2(\Omega))} \le Ch^k \tag{11}$$

where $C = C(\|u\|_{L^2([0,T],H^{k+1})}, \|u_t\|_{L^2([0,T],H^{k+1})}, \Omega, T, u_0)$.

Theorem 3.1. *If the conditions in this section are satisfied the mean error $\overline{e}$ and discrete mean error $\overline{e}^h$ converge optimally,*

$$\begin{aligned}\|\overline{e}^h\|_{L^\infty(0,T,L^2(\Omega))} &\le C((h + h^{1.5})\|e\|_{L^\infty(0,T,L^2(\Omega))} + \|e(0)\|)\\ &\le C_1(h^{k+1} + h^{k+1.5})\end{aligned} \tag{12}$$

and

$$\begin{aligned}\|\overline{e}\|_{L^\infty(0,T,L^2(\Omega))} &\le C((h + h^{1.5} + h^2)\|e\|_{L^\infty(0,T,L^2(\Omega))} + \|e(0)\|)\\ &\le C_1(h^{k+1} + h^{k+1.5} + h^{k+2}).\end{aligned}$$

In the above inequalities we have that $C = C(\Omega, T, |\vec{a}|_\infty)$ and $C_1 = C_1(u, \Omega, T, |\vec{a}|_\infty)$.

Proof. The error equation is

$$(e_t, v_h) + (\vec{a}\cdot\nabla e, v_h) = 0 \quad \text{for all } v_h \in X_h.$$

We may therefore set in the above equation $v_h = \overline{\overline{e}^h}^h \in X_h$ and we get

$$(e_t, \overline{\overline{e}^h}^h) + (\vec{a}\cdot\nabla e, \overline{\overline{e}^h}^h) = 0. \tag{13}$$

We have that

$$(e_t, \overline{\overline{e}^h}^h) = (\overline{(e_t)}^h, \overline{e}^h) = ((\overline{e}^h)_t, \overline{e}^h) = \frac{1}{2}\frac{d}{dt}(\overline{e}^h, \overline{e}^h) = \frac{1}{2}\frac{d}{dt}\|\overline{e}^h\|^2.$$

The first equality is due to the symmetry of the discrete filtering operator $\mathcal{S}_h$ with respect to the L^2 inner product on Ω, see Remark 2.1, and the second equality is valid because the discrete filtering operator commutes with the time derivative, i.e. $\overline{(e_t)}^h = (\overline{e}^h)_t$.

Using Eq. (13) we obtain

$$\frac{1}{2}\frac{d}{dt}\|\overline{e}^h\|^2 = -(\vec{a}\cdot\nabla e, \overline{\overline{e}^h}^h). \tag{14}$$

But, because in the periodic setting the filtering operator $u \to \overline{u}$ and differentiation commute, we have that

$$(\vec{a}\cdot\nabla e, \overline{\overline{e}}) = (\overline{\vec{a}\cdot\nabla e}, \overline{e}) = (\vec{a}\cdot\nabla\overline{e}, \overline{e}) = 0.$$

We may therefore add the above term to the right hand side of equality (14) and get

$$\frac{1}{2}\frac{d}{dt}\|\overline{e}^h\|^2 = (\vec{a}\cdot\nabla e, \overline{\overline{e}} - \overline{\overline{e}^h}^h).$$

We add and subtract $\overline{\overline{e}^h}$ on the right hand side term

$$\frac{1}{2}\frac{d}{dt}\|\overline{e}^h\|^2 = (\vec{a}\cdot\nabla e, \overline{\overline{e}} - \overline{\overline{e}^h} + \overline{\overline{e}^h} - \overline{\overline{e}^h}^h) = (\vec{a}\cdot\nabla e, \overline{\overline{e}} - \overline{\overline{e}^h}) + (\vec{a}\cdot\nabla e, \overline{\overline{e}^h} - \overline{\overline{e}^h}^h). \tag{15}$$

The two terms on the right hand side of (15) are treated separately.

$$\begin{aligned}(\vec{a}\cdot\nabla e, \bar{\bar{e}} - \overline{\bar{e}^h}) &= (\vec{a}\cdot\nabla e, \overline{\bar{e} - \bar{e}^h}) = (\vec{a}\cdot\nabla\bar{e}, \bar{e} - \bar{e}^h) = -(\bar{e}, \vec{a}\cdot\nabla(\bar{e} - \bar{e}^h)) \\ &\le |\vec{a}|_\infty \|\bar{e}\| \|\nabla(\bar{e} - \bar{e}^h)\| \le Ch|\vec{a}|_\infty \|\bar{e}\| \|e\| \end{aligned} \tag{16}$$

where $C = C(\Omega)$. For simplicity, we will use this general constant C through the proof, although it might change its value from one inequality to another, but the parameters it depends on will be indicated.

In the last inequality we also used inequality (7) in Remark 2.2, i.e.

$$\|\nabla(\bar{e} - \bar{e}^h)\| \le Ch\|e\|.$$

But, again using inequality (7) we have that

$$\|\bar{e}\| \le \|\bar{e}^h\| + Ch^2\|e\|$$

and so the estimate becomes

$$(\vec{a}\cdot\nabla e, \bar{\bar{e}} - \overline{\bar{e}^h}) \le Ch\|\bar{e}^h\|\|e\| + Ch^3\|e\|^2$$

where $C = C(\Omega, |\vec{a}|_\infty)$. We now estimate the second term in (15).

$$(\vec{a}\cdot\nabla e, \overline{\bar{e}^h} - \overline{\bar{e}^h}^h) = -(e, \vec{a}\cdot\nabla(\overline{\bar{e}^h} - \overline{\bar{e}^h}^h)) \le |\vec{a}|_\infty \|e\| \|\nabla(\overline{\bar{e}^h} - \overline{\bar{e}^h}^h)\| \le Ch\|e\|\|\bar{e}^h\|$$

where $C = C(\Omega, |\vec{a}|_\infty)$. Here again, we have used inequality (7) to get

$$\|\nabla(\overline{\bar{e}^h} - \overline{\bar{e}^h}^h)\| \le Ch\|\bar{e}^h\|.$$

We therefore have

$$\frac{1}{2}\frac{d}{dt}\|\bar{e}^h\|^2 \le Ch\|\bar{e}^h\|\|e\| + Ch^3\|e\|^2 \le \|\bar{e}^h\|^2 + C(h^2 + h^3)\|e\|^2. \tag{17}$$

In the last inequality we have used the scalar inequality $2\alpha\beta \le \alpha^2 + \beta^2$ to bound the term $Ch\|\bar{e}^h\|\|e\|$. An application of Gronwall's inequality gives

$$\|\bar{e}^h\|^2_{L^\infty(0,T,L^2(\Omega))} \le C((h^2 + h^3)\|e\|^2_{L^\infty(0,T,L^2(\Omega))} + \|\bar{e}^h(0)\|^2) \tag{18}$$

with $C = C(\Omega, T, |\vec{a}|_\infty)$.

Next, due to the stability of the discrete differential filter, see Remark 2.1, we have that

$$\|\bar{e}^h(0)\| \le \|e(0)\|$$

so that inequality (18) becomes

$$\|\bar{e}^h\|_{L^\infty(0,T,L^2(\Omega))} \le C((h + h^{1.5})\|e\|_{L^\infty(0,T,L^2(\Omega))} + \|e(0)\|). \tag{19}$$

To obtain the order of convergence of the above right term we use inequality (11) together with the estimate

$$\|e(0)\| = \|u(0) - \bar{u}^h(0)\| \le Ch^{k+1}|u(0)|_{k+1} \tag{20}$$

to obtain

$$\|\bar{e}^h\|_{L^\infty(0,T,L^2(\Omega))} \le C_1(h^{k+1} + h^{k+1.5})$$

where $C_1 = C_1(u, \Omega, T, |\vec{a}|_\infty)$.

Using the triangle inequality and inequalities (7) and (19) we get

$$\begin{aligned}\|\bar{e}\|_{L^\infty(0,T,L^2(\Omega))} &\le \|\bar{e}^h\|_{L^\infty(0,T,L^2(\Omega))} + \|\bar{e} - \bar{e}^h\|_{L^\infty(0,T,L^2(\Omega))} \\ &\le \|\bar{e}^h\|_{L^\infty(0,T,L^2(\Omega))} + Ch^2\|e\|_{L^\infty(0,T,L^2(\Omega))} \\ &\le C((h + h^{1.5} + h^2)\|e\|_{L^\infty(0,T,L^2(\Omega))} + \|e(0)\|)\end{aligned}$$

where $C = C(\Omega, T, |\vec{a}|_\infty)$.

To extract the order of convergence we use inequalities (11) and (20) to get

$$\|\bar{e}\|_{L^\infty(0,T,L^2(\Omega))} \le C_1(h^{k+1} + h^{k+1.5} + h^{k+2})$$

where $C_1 = C_1(u, \Omega, T, |\vec{a}|_\infty)$. $\square$

4. The optimal scheme

We will assume that the solution u of problem (9) satisfies the regularity conditions $u \in C^1([0,T], H^3_{\#}(\Omega))$ and $u, u_t \in L^2([0,T], H^{k+3}_{\#}(\Omega))$. It follows that the function $w = u - \Delta u$ is the solution of the advection problem

$$w_t + \vec{a} \cdot \nabla w = f - \Delta f$$

with initial condition $w_0 = u_0 - \Delta u_0$, subject to periodic boundary condition.

We denote by $w^h \in C^1([0,T], X)$ its finite element approximation satisfying

$$\left(\frac{dw_h}{dt}, v_h\right) + (\vec{a} \cdot \nabla u_h, v_h) = (f, v_h) + (\nabla f, \nabla v_h), \quad \text{for all } v_h \in X_h,$$

with the initial condition $w_h(0)$ being an interpolant of $w(0)$ satisfying the approximation property, see inequality (4).

We next prove that $\overline{w_h}^h$, i.e. the discrete filter of w_h, is an optimal approximation of u.

The error $\|u - \overline{w_h}^h\|$ can be estimated as follows:

$$\|u - \overline{w_h}^h\| = \|\overline{w} - \overline{w_h}^h\| \le \|\overline{w} - \overline{w}^h\| + \|\overline{w}^h - \overline{w_h}^h\|.$$

The first term is estimated using inequality (8)

$$\|\overline{w} - \overline{w}^h\| \le Ch^{k+1}\|\overline{w}\|_{k+1} = Ch^{k+1}\|u\|_{k+1}$$

whereas the second is estimated using the first estimate (12) in the previous theorem

$$\|\overline{w}^h - \overline{w_h}^h\| \le C_1(h^{k+1} + h^{k+1.5})$$

where $C_1 = C_1(u, \Omega, T, |\vec{a}|_\infty)$.

Collecting terms gives the result in the next theorem.

Theorem 4.1. *With w_h constructed as above there holds*

$$\|u - \overline{w_h}^h\|_{L^\infty(0,T;L^2(\Omega))} \le C_1(h^{k+1} + h^{k+1.5}) = \mathcal{O}(h^{k+1})$$

where $C_1 = C_1(u, \Omega, T, |\vec{a}|_\infty)$ and does not depend on h.

5. Numerical studies

In this section we use the classical FEM scheme and the proposed optimal scheme described in the previous section to solve an advection problem and check the rates obtained in the previous section.

The problem has been solved numerically with the FreeFEM package, [12,13].

We have considered the advection equation

$$u_t + 0.1u_x = f$$

on the rectangle $\Omega = [0,1] \times [0, 0.25]$ with exact solution

$$u_{exact}(t,x,y) = \sin(t)\sin^4(\pi x)\sin^4(4\pi y)$$

and corresponding

$$f(t,x,y) = \cos(t)\sin^4(\pi x)\sin^4(4\pi y) + 0.1 \cdot 4\pi \sin(t)\sin^3(\pi x)\cos(\pi x)\sin^4(4\pi y).$$

We generate an initial unstructured mesh with 10 evenly placed nodes on the $y = 0$ boundary, 4 even nodes on the $x = 1$ boundary, 14 even nodes on the $y = 0.25$ boundary and 6 even nodes on the $x = 0$ boundary.

This mesh is then successively refined using the FreeFEM function trunc by splitting each side of each triangle in the mesh into 2, 4, 8, 16, 32, 64 equal parts. These meshes are the computational meshes on which the problem will be solved (see Fig. 1).

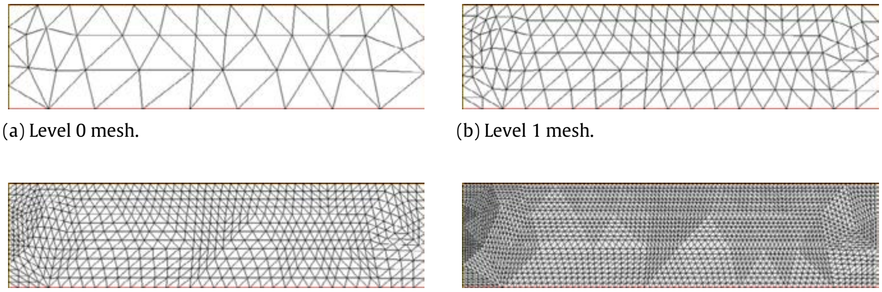
For the classical FEM scheme the Crank–Nicolson time discretization is used to solve the advection equation

$$\frac{1}{\Delta t}(u_h^{n+1} - u_h^n, v_h) + 0.1 \cdot \frac{1}{2}(u_{h,x}^{n+1} + u_{h,x}^n, v_h) = \frac{1}{2}(f^{n+1} + f^n, v_h)$$

for every $v_h \in X_h$ with $u_h^0 = 0$ and zero boundary conditions. Here $f^n(x,y) = f(t_n, x, y)$, $t_n = n\Delta t$.

For the proposed scheme, one first solves for w_h^{n+1}

$$\frac{1}{\Delta t}(w_h^{n+1} - w_h^n, v_h) + 0.1 \cdot \frac{1}{2}(w_{h,x}^{n+1} + w_{h,x}^n, v_h) = \frac{1}{2}(f^{n+1} + f^n, v_h) + \frac{1}{2}(\nabla(f^{n+1} + f^n), \nabla v_h)$$



(a) Level 0 mesh.

(b) Level 1 mesh.

(b) Level 2 mesh.

(b) Level 3 mesh.

Fig. 1. The initial mesh and the first three computational meshes.

Table 1
$L^\infty(L^2)$ errors and rates for the usual FEM and the proposed scheme, 1000 time iterations, $\Delta t = 0.005$. The predicted rate order of the proposed scheme is 3.

Mesh size	Usual FEM		Proposed scheme	
h	$L^\infty(L^2)$ error	Rate	$L^\infty(L^2)$ error	Rate
0.0768796	0.000669005		0.000655676	
0.0384398	9.0793e−005	2.881	8.4855e−005	2.949
0.0192199	1.3571e−005	2.742	1.0728e−005	2.983
0.00960995	2.4977e−006	2.441	1.3463e−006	2.994
0.00480498	5.5426e−007	2.172	1.6867e−007	2.996
0.00240249	1.3382e−007	2.050	2.1207e−008	2.991

for every $v_h \in X_h$ with $w_h^0 = 0$ and zero boundary conditions and at each step one computes $\overline{w_h^{n+1}}^h$, the discrete mean of w_h^{n+1},

$$(\overline{w_h^{n+1}}^h, v_h) + (\nabla \overline{w_h^{n+1}}^h, \nabla v_h) = (w_h^{n+1}, v_h)$$

for every $v_h \in X_h$ with zero boundary conditions.

$\overline{w_h^{n+1}}^h$ will be the approximation of the exact solution $u^{n+1} = u(t_{n+1})$ provided by the proposed scheme.

The two schemes use $P2$ finite elements on the six computational meshed described before, therefore the predicted order rate for the classical FEM will be 2, whereas for the proposed scheme will be 3, (see Table 1).

In the numerical test $\Delta t = 0.0005$ and the number of time iterations is 1000. If one halves the time step (i.e. set $\Delta t = 0.00025$) and doubles the total number of time steps (i.e. number of time steps equals 2000) the rates in Table 1 will not change significantly.

6. Conclusions

The scheme presented herein solves the model advection Eq. (9) optimally on unstructured quasi-uniform grids and for general elements assuming periodic boundary conditions and a constant advection field. We also assume a bit more regularity (two units) than the one required by the classical suboptimal theory.

Extension of these results to other types of boundary conditions may be considered in the future as well as the performance of this method in the context of problems with non-smooth solutions where the classical FEM might exhibit spurious oscillations in the vicinity of sharp layers. It would also be interesting to see this scheme applied to related convection–diffusion problems.

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