NEW NUMERICAL MODEL AND TECHNIQUE FOR WATERHAMMER

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ABSTRACT: How efficiently a second-order model is handled is analysed for simplified equations governing waterhammer appearance in piping systems. A series solution method and a Newton-Raphson method with new calculation steps are proposed, which are efficient numerical methods, through omitting trivial terms computed within a truncation error. With fewer calculations than required previously, the proposed methods can give a solution with a required accuracy without any iteration. The second-order model, therefore, offers more accurate and efficient methods than those offered by a first-order model. However, the second-order model causes an error in computing steady flows; ways to remove or reduce this error are shown. The validity of the analyses is examined by numerical computations in which system parameters are varied over a wide range of parameters.

INTRODUCTION

In predicting and controlling waterhammer and transients, it is required to precisely compute the behavior of pipeline systems using methods which reduce the time for calculations as much as possible. In spite of the importance of this problem, this issue has not been investigated sufficiently.

The method of characteristics is mainly employed to solve hydraulic problems in piping systems. The method of specified time intervals (5) and the characteristic-grid method (2) are known as solution procedures for the basic equations. For the simplified equations, in which smaller terms are neglected, first and second-order models (2, 4, 6, 7, and 8) are used to solve transients problems in relative stiff pipes. When viscous effects are very important, a first-order model may give an incorrect answer and, in extreme cases, cause a numerical instability in the solution. In order to improve the accuracy of the solution without using smaller time steps, a second-order model is applied. A first-order model gives an explicit solution while a second-order model usually gives an implicit solution which is obtained after a few iterations.

Therefore, for computational purposes, at each grid point a second-order model would need several calculations, as much as would be required in a first-order model. However, if a second-order model can yield a solution with nearly the same amount of calculations as a first-order model requires, the former is more efficient than the latter.

The object of this paper is to answer the question, "How efficiently can a solution with the required accuracy be obtained by a second-order model?" Methods for handling a second-order model without any iter-

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Note.—Discussion open until November 1, 1984. To extend the closing date one month, a written request must be filed with the ASCE Manager of Technical and Professional Publications. The manuscript for this paper was submitted for review and possible publication on January 4, 1983. This paper is part of the *Journal of Hydraulic Engineering*, Vol. 110, No. 6, June, 1984. ©ASCE, ISSN 0733-9429/84/0006-0736/$01.00. Paper No. 18941.

ation are analyzed. First, a truncation error due to discretization in deriving finite-difference equations is estimated using a small parameter related to the product of a friction factor and a time step size. The terms smaller than the truncation error are omitted, and two new numerical methods (a series solution method and a Newton-Raphson method with new calculation steps) are proposed as efficient solution procedures, which are applicable not only to a system with high friction losses but also to a system with moderately high friction losses. However, it is shown that the methods presented cause an error due to an inadequate estimate of a steady flow. Ways to remove or reduce this error will be shown.

The validity of the analyses is examined by numerical computations in which the system parameters are varied widely.

Basic Equations

Under a condition that the velocity in a pipe, V, is much smaller than a wavespeed of pressure pulse, a^*, the basic equations for momentum and mass are:

$$\frac{\partial V}{\partial T} + g\frac{\partial H}{\partial X} + FV|V|^m = 0 \quad (1)$$

and $$\frac{\partial H}{\partial T} + \frac{a^{*2}}{g}\frac{\partial V}{\partial X} = 0 \quad (2)$$

For example, if the Darcy-Weisbach formula is applied to the friction term, we have

$$F = \frac{f}{2D} \quad (3)$$

$$m = 1 \quad (4)$$

in which V = average flow velocity (m/s); H = head above the datum line (m); D = diameter of the pipe (m); F = friction factor; f = Darcy-Weisbach friction coefficient; g = gravitational acceleration (m/s^2); m = exponent of the power in the friction term; and X = distance along the pipe (m), and T = time (sec).

Characteristic Equations.—Transforming Eqs. 1 and 2 into dimensionless characteristic forms, we have:

$$C^+: \quad \frac{dx}{dt} = 2 \quad (5)$$

$$dh + B\,dv + 2\sigma v|v|^m dt = 0 \quad (6)$$

$$C^-: \quad \frac{dx}{dt} = -2 \quad (7)$$

$$-dh + B\,dv + 2\sigma v|v|^m dt = 0 \quad (8)$$

in which C^+ and C^- = symbols for advancing and receding characteristic lines; V_0 = characteristic velocity, H_0 = characteristic head; L_0 = char-

J. Hydraul. Eng. 1984.110:736-748.

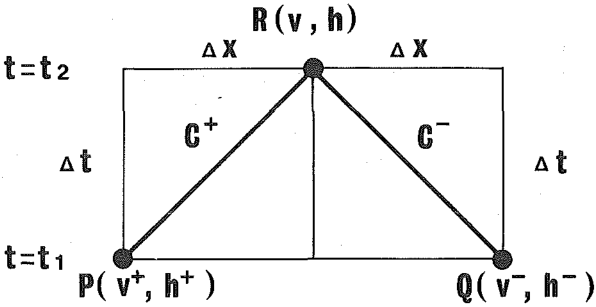

FIG. 1.—Time and Space Grid Model for Computation at Interior Points

acteristic length; $v = V/V_0$, $h = H/H_0$; $t = T/(2L_0/a*)$; $x = X/L_0$, $B = (a* V_0/gH_0)$; and $\sigma = F V_0^{m+1} L/(gH_0)$.

Finite-Difference Equations.—A time and space grid model for computation is shown in Fig. 1. The velocities, v^+ and v^-, and the heads, h^+ and h^-, at points P and Q are known while v and h at point R are unknown.

Eq. 9 is a first-order approximation to integration of the friction term

$$\int_{t_1}^{t_2} v|v|^m dt \simeq v(t_1)|v(t_1)|^m (t_2 - t_1) \quad (9)$$

It is already known that the first-order model may be inadequate to deal with unsteady flow situations in pipeline systems with high friction losses. Thus, in order to improve the accuracy Eq. 10 has been used as a second-order approximation to the integration

$$\int_{t_1}^{t_2} v|v|^m dt \simeq \left[\frac{v(t_1) + v(t_2)}{2}\right] \left|\frac{v(t_1) + v(t_2)}{2}\right|^m (t_2 - t_1) \quad (10)$$

From Eqs. 6, 8, and 10, we obtain the following finite-difference equations which are usually solved by the Newton-Raphson method

$$C^+: \quad (h - h^+) + B(v - v^+) + \epsilon(v + v^+)|v + v^+|^m = 0 \quad (11)$$

$$C^-: \quad -(h - h^-) + B(v - v^-) + \epsilon(v + v^-)|v + v^-|^m = 0 \quad (12)$$

in which $\epsilon = \sigma \Delta t/2^m$ and $\Delta t = t_2 - t_1$.

Newton-Raphson Method.—The elimination of h from Eqs. 11 and 12 leads to an algebraic equation for the unknown velocity v:

$$h^- - h^+ + B(2v - v^- - v^+) + \epsilon[(v + v^+)|v + v^+|^m + (v + v^-)|v + v^-|^m] = 0 \quad (13)$$

In general the left-hand side of Eq. 13 does not become zero for an assumed value of v in the iterative process. Denoting the error corresponding to the left-hand side of Eq. 13 by "E," the derivative of E with respect to v gives

$$\frac{dE}{dv} = 2B + \epsilon(m + 1)[|v + v^+|^m + |v + v^-|^m] \quad (14)$$

$(v^+ + v^-)/2$ is usually used as the first estimate of v. If a criterion to end the iterative process is known, the iterations are carried out at each grid point until the criterion is satisfied by replacing the value v given before by $v + \Delta v$:

$$\Delta v = -\frac{E}{\dfrac{dE}{dv}} \quad \text{(15)}$$

in which Δv denotes a correction term.

The incompleteness of this method is that the criterion is not sufficiently clear. One can not predict how many times the calculations should be repeated before reaching the required accuracy.

Truncation Error.—The proposed new explicit method will save calculation time by disregarding smaller terms within a truncation error. The truncation error due to the second-order approximation is estimated using Crandall's method (1).

Assuming that the values of v are positive (or negative) between P and R or between Q or R, the velocity in the friction term is expanded at point P or Q in the Taylor's series:

$$v(t) = v(t_1 + y \cdot \Delta t) = v_0(t_1) + v_1(t_1) \cdot y \cdot \Delta t + v_2(t_1) \frac{(y \cdot \Delta t)^2}{2} + \ldots \quad \text{(16)}$$

in which y, given by Eq. 17, = a parameter expressing characteristic lines along which the integration is formed:

$$t = t_1 + y \cdot \Delta t \quad \text{(17)}$$

in which $0 \leqq y \leqq 1$. In addition, $v_k(t_1)$ is kth differential coefficient of v with respect to $(y \cdot \Delta t)$.

The transformation of the parameter in the left-hand side of Eq. 18 from t to y leads to the right-hand side:

$$\int_{t_1}^{t_2} v^{m+1}(t)\, dt = \Delta t \cdot \int_0^1 v^{m+1}(t_1 + y \cdot \Delta t)\, dy \quad \text{(18)}$$

The integrand in the right-hand side of Eq. 18 becomes

$$v^{m+1}(t_1 + y \cdot \Delta t) = v_0^{m+1} \left\{ 1 + (m+1) \frac{v_1}{v_0} y \cdot \Delta t + \left[m(m+1)\left(\frac{v_1}{v_0}\right)^2 + (m+1)\frac{v_2}{v_0} \right] \frac{(y \cdot \Delta t)^2}{2} + \ldots \right\} \quad \text{(19)}$$

in which $y \cdot \Delta t << 1$, and v_k denotes $v_k(t_1)$ for simplicity. Substitution of Eq. 19 into Eq. 18 yields

$$\Delta t \cdot \int_0^1 v^{m+1}(t_1 + y \cdot \Delta t)\, dy = \Delta t\, v_0^{m+1} \left\{ 1 + \frac{(m+1)}{2} \frac{v_1}{v_0} \Delta t + (m+1)\left[m\left(\frac{v_1}{v_0}\right)^2 + \frac{v_2}{v_0} \right] \frac{(\Delta t)^2}{6} + \ldots \right\} \quad \text{(20)}$$

The right-hand side of Eq. 10 becomes

$$\Delta t \cdot \left[\frac{v(t_1) + v(t_2)}{2}\right]^{m+1} = v_0^{m+1}\left\{1 + \frac{(m+1)}{2} \cdot \frac{v_1}{b_0} \cdot \Delta t\right.$$

$$\left. + \frac{(m+1)}{2}\left[\frac{m}{2} \cdot \left(\frac{v_1}{v_0}\right)^2 + \frac{v_2}{v_0}\right]\frac{(\Delta t)^2}{2} + \ldots\right\} \cdot \Delta t \quad (21)$$

in which $v(t_2) = v(t_1 + \Delta t)$. Using Eqs. 10, 18, 20, and 21, the truncation error is estimated as follows

$$\int_{t_1}^{t_2} v(t)^{m+1} dt - \left[\frac{v(t_1) + v(t_2)}{2}\right]^{m+1} (t_2 - t_1)$$

$$\simeq v_0^{m+1}\left[\frac{m(m+1)}{2}\left(\frac{v_1}{v_0}\right)^2 - (m+1)\frac{v_2}{v_0}\right]\frac{(\Delta t)^3}{12} + O(\Delta t^4) \quad (22)$$

Since the truncation error is of the same order as Δt^3 or ϵ^3, i.e., $O(\epsilon^3)$, it follows that the left-hand side of Eq. 13 may be different from the value of zero in $O(\epsilon^3)$.

New Explicit Method (Series Solution Method)

The solution of Eq. 13 containing the truncation error can be obtained explicitly. Unknowns, v and h, at point R can be expanded into power series of the small parameter ϵ as shown in Eqs. 23 and 24, because v and h must include ϵ implicitly. The terms smaller than $O(\epsilon^3)$ are neglected in the following analysis.

$$v = \tilde{u}_0 + \epsilon\,\tilde{u}_1 + \epsilon^2\tilde{u}_2 + O(\epsilon^3) \quad (23)$$

$$h = \tilde{h}_0 + \epsilon\tilde{h}_1 + \epsilon^2\tilde{h}_2 + O(\epsilon^3) \quad (24)$$

where the condition that $\epsilon << 1$ is needed for convergence of the solution of Eqs. 23 and 24, and $\tilde{u}_k$ and $\tilde{h}_k$ are kth component of the power series of ϵ for v and h, respectively. The substitution of the solution, which is shown by Eqs. 23 and 24, into Eqs. 11 and 12 yields Eqs. 25, 26, and 27 for velocity v.

$$\tilde{u}_0 = \frac{\dfrac{v^+ + v^- + (h^+ - h^-)}{B}}{2} \quad (25)$$

$$\tilde{u}_1 = -\frac{(\tilde{u}_0 + v^+)|\tilde{u}_0 + v^+|^m + (\tilde{u}_0 + v^-)|\tilde{u}_0 + v^-|^m}{2B} \quad (26)$$

$$\tilde{u}_2 = -(m+1)(|\tilde{u}_0 + v^+|^m + |\tilde{u}_0 + v^-|^m)\frac{\tilde{u}_1}{2B} \quad (27)$$

Thus, the new explicit method is formulated as:

1. From Eqs. 25–27, $\tilde{u}_0$, $\tilde{u}_1$, and $\tilde{u}_2$ are calculated by use of the four knowns, v^+, v^-, h^+, and h^-, at points P and Q.
2. Using Eq. 24, an unknown velocity v at point R becomes known.

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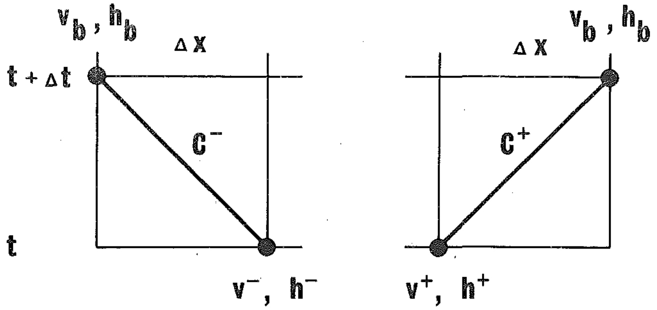

FIG. 2.—Grid Model for Computation at Boundary Points (Left: at Upstream Boundary, Right: at Downstream Boundary)

3. When v becomes known, either Eq. 11 or Eq. 12 can be used to find an unknown head h.

Calculation at Boundary Points.—Now, we will describe the calculation technique at boundary points by the series solution method. Fig. 2 shows grid models at boundary points.

Eq. 12 holds along C^- for the upstream end (see Fig. 2). If the velocity v at the boundary is given by v_b, we obtain

$$h_b = h^- + B(v_b - v^-) + \epsilon(v_b + v^-)|v_b + v^-|^m \quad (28)$$

and if h is given by h_b, in the same way as the calculation at interior points, we get

$$v_b = \tilde{u}_0 + \epsilon\tilde{u}_1 + \epsilon^2\tilde{u}_2; \quad \tilde{u}_0 = \frac{h_b - h^-}{B} + v^-;$$

$$\tilde{u}_1 = -(\tilde{u}_0 + v^-)\frac{|\tilde{u}_0 + v^-|^m}{B}; \quad \tilde{u}_2 = -(m+1)|\tilde{u}_0 + v^-|^m\frac{\tilde{u}_1}{B} \quad (29)$$

Eq. 11 holds along C^+ for the downstream end. If the velocity at the boundary, v, is given by v_b, we have

$$h_b = h^+ - B(v_b - v^+) - \epsilon(v_b + v^+)|v_b + v^+|^m \quad (30)$$

and if the head, h, is given by h_b, we obtain

$$v_b = \tilde{u}_0 + \epsilon\tilde{u}_1 + \epsilon^2\tilde{u}_2; \quad \tilde{u}_0 = \frac{h^+ - h_b}{B} + v^+;$$

$$\tilde{u}_1 = -(\tilde{u}_0 + v^+)\frac{|\tilde{u}_0 + v^+|^m}{B}; \quad \tilde{u}_2 = -(m+1)|\tilde{u}_0 + v^+|^m\frac{\tilde{u}_1}{B} \quad (31)$$

Calculation of Steady Flow.—Velocities in a steady flow are known along C^+ or C^-. Therefore, it should be noted that a theoretical solution of a steady flow does not contain any truncation error, since the friction term can be integrated without any approximations. Consequently, an implicit solution of Eq. 13 can be expressed in an infinite series of ϵ. If the series solution method, which is valid for transients, is applied to a

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steady flow, the following error is produced: By putting $v^+ = v^- = 1$ and $h^+ - h^- = 2\sigma/M$, we have

$$\delta = v - 1 = \frac{\epsilon^3}{B^3}\left(80 + 192\frac{\epsilon}{B} + 128\frac{\epsilon^2}{B^2}\right) \qquad (32)$$

in which δ = the order of the error estimated for a steady flow, and M = the number of reaches.

Considering that $\epsilon = \sigma \cdot \Delta t/2^m$ and $\Delta t = 1/(2M)$ and assuming that the exponent $m = 1$, we have

$$\epsilon = \frac{\sigma}{4M} \qquad (33)$$

$$\delta = 1.25\,Y^3 + 0.75\,Y^4 + 0.125\,Y^5 \qquad (34)$$

in which $Y = \sigma/(BM)$.

The series solution method may yield an unstable solution due to the inadequate estimate of a steady flow if Y is taken to be so large that δ approaches $O(10^{-1})$. Then, the adequate number of reaches, M, should be selected from Eq. 35 so that δ becomes less than $O(10^{-3})$:

$$M = \frac{\sigma}{(BY)} \qquad (35)$$

Although this criterion is satisfied with a small number of reaches in usual cases, a larger number of reaches is required to avoid a numerical instability in calculations of a system with a high friction loss. In that case, a Newton-Raphson method with improved calculation steps should be used.

New Formulation of Newton-Raphson Method.—We will show that the Newton-Raphson method does not need any iterative process if an initial value of the unknown velocity, v, is properly estimated. To do this, the iterative process has to be analysed using the algorithm. The substitution of $(v^+ + v^-)/2$ into Eq. 15 leads to the first correction Δv^0:

$$\Delta v^0 = -\frac{(h^+ - h^-)}{2B} + O(\epsilon) \qquad (36)$$

From this, it is clear that the nonviscous solution, $\tilde{u}_0$, should be used as the first estimate of v, because the first iteration gives only $\tilde{u}_0$. Then, we have

$$\Delta v^0 = -\frac{\epsilon}{2B}[(\tilde{u}_0 + v^+)|\tilde{u}_0 + v^+|^m + (\tilde{u}_0 + v^-)|\tilde{u}_0 + v^-|^m]$$

$$+ \frac{(m+1)\,\epsilon^2}{4B^2}(|\tilde{u}_0 + v^+|^m + |\tilde{u}_0 + v^-|^m)[(\tilde{u}_0 + v^+)|\tilde{u}_0 + v^+|^m$$

$$+ (\tilde{u}_0 + v^-)|\tilde{u}_0 + v^-|^m] + O(\epsilon^3) \qquad (37)$$

The second correction term is given by

$$\Delta v^1 = O(\epsilon^3) \qquad (38)$$

The first and second terms in the right-hand side of Eq. 37 correspond

TABLE 1.—Comparison Among Solution Procedures in Terms of Efficiency in Calculations

Solution procedure (1)	Addition and subtraction (2)	Multiplication and division (3)	Exponentiation calculation (4)
Series solution method	9	7	2
Newton-Raphson method	11	9	2
First-order model	5	5	2

to the second and third terms in the right-hand side of Eq. 23. This shows that the Newton-Raphson method guarantees a second order convergence, which means that each iteration near the true solution yields a solution with a higher accuracy by the square of ϵ than that obtained before. Therefore, when the nonviscous solution, $\tilde{u}_0$, is used as an initial estimate of v, any iteration is not required to obtain the solution with the accuracy of $O(\epsilon^3)$.

The new calculation steps of the Newton-Raphson method are: (1) Calculate the inviscid solution $\tilde{u}_0$ from Eq. 25; (2) calculate the terms, $(-E)$ and (dE/dv), from Eqs. 13 and 14, and divide the former by the latter and, as the result, the correction Δv is obtained; (3) let v be $(\tilde{u}_0 + \Delta v)$; and (4) the unknown, h, is found by substituting v into Eq. 11 or Eq. 12.

The Newton-Raphson method also does not hold for a steady flow. The error, δ, which is caused in calculating the steady flow by the Newton-Raphson method, can also be estimated using Eq. 32 as with the series solution method. Consequently, we have

$$\delta = v - 1 = \frac{\epsilon^3}{B^3}\left(16 - 64\frac{\epsilon}{B} + 128\frac{\epsilon^2}{B^2}\right) + O(\epsilon^6) \quad \text{(39)}$$

It is clear from the comparison between Eq. 32 and Eq. 39 that the error with the Newton-Raphson method becomes much smaller than that with the series solution method.

Next, the efficiency in calculations is compared among three solution procedures (the series solution method, the Newton-Raphson method, and the explicit method for the first-order model) by counting the minimum number of times of operations (addition, subtraction, multiplication, division, and exponentiation calculation), which are required for obtaining an unknown velocity v at each grid point. The result is shown in Table 1, from which we can find that each solution procedure has the same number of exponentiation calculation times (twice times). Therefore, it follows that at each grid point the work for solving v by the second-order model is almost the same as that by the first-order model, because the exponentiation calculation consumes much more time than other kinds of operations.

Numerical Results

In the following, numerical computations will be used to examine the usefulness and limitation of the series solution method and the Newton-

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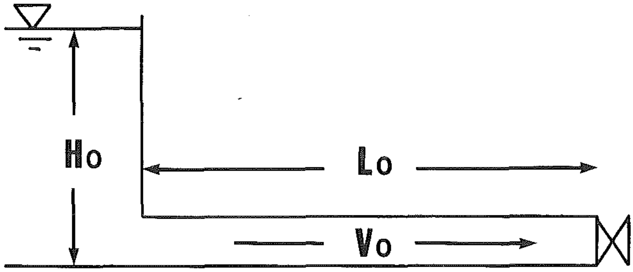

FIG. 3.—Simple Piping System

Raphson method with the new calculation steps. The system shown in Fig. 3 is a model for computation, which is composed of a constant-head reservoir at the upstream end, a control valve at the downstream end, and a single pipe leading from the reservoir to the valve.

Waterhammer and transients are caused by closing the valve, which begins to move at time zero and is completely closed in a dimensionless time t_c. The velocity at the valve varies linearly with time during that time interval. The exponent, m, in the friction term, is assumed to be 1.0. Then, an unknown velocity v can be explicitly determined using Eq. 13 as roots of quadratic equations and solutions of linear equations. The solution procedures using the roots and the solutions is denoted by the "algebraic solution" method, which is available for comparisons with other solution methods of the second-order model, since the "algebraic solution" method holds for a steady flow.

Each calculation begins with an initial steady condition set at time zero, and then is carried forward using the algorithm for each solution procedure.

Case 1.—The system parameters and the valve closure time are: B = 1.0 and 2.0; σ = 0.2, 0.4, and 0.8; and t_c = 1.0 and 5.0. The numbers of reaches, M, are determined from Eq. 35 so that δ becomes 10^{-3}, 10^{-4}, and 10^{-5}. In Tables 2 through 4, the numerical results are compared with the most probable values calculated with M = 100 in terms of the maximum waterhammer head. It is clear that, even for systems with mod-

TABLE 2.—Maximum Waterhammer Head Computed

t_c (1)	B (2)	σ (3)	M (4)	Series solution method (5)	Newton-Raphson method (6)	"Algebraic Solution" method (7)	First-order model (8)
1.0	1.0	0.2	2	1.93717	1.93447	1.93391	1.87111
			5	1.92972	1.92926	1.92915	1.90701
			10	1.92861	1.92849	1.92846	1.91789
			100	1.92823	1.92823	1.92823	1.92722
	2.0	0.2	2	2.93654	2.93583	2.93567	2.87309
			5	2.93168	2.93155	2.93152	2.90948
			10	2.93097	2.93093	2.93092	2.92041
			100	2.93073	2.93073	2.93073	2.92972

TABLE 3.—Maximum Waterhammer Head Computed

t_c (1)	B (2)	σ (3)	M (4)	Series solution method (5)	Newton-Raphson method (6)	"Algebraic Solution" method (7)	First-order model (8)
1.0	1.0	0.4	4	1.85643	1.85169	1.85073	1.79316
			10	1.84863	1.84788	1.84771	1.82622
			20	1.84751	1.84732	1.84728	1.83679
			100	1.84715	1.84714	1.84714	1.84509
		0.8	8	1.67082	1.66537	1.66428	1.60694
			19	1.66341	1.66249	1.66228	1.63901
			40	1.66220	1.66199	1.66194	1.65105
			100	1.66190	1.66187	1.66186	1.65756
	2.0	0.4	2	2.87433	2.86893	2.86782	2.74222
			5	2.85945	2.85851	2.85830	2.81402
			10	2.85721	2.85697	2.85692	2.83578
			100	2.85646	2.85646	2.85646	2.85444
		0.8	4	2.71287	2.70339	2.70136	2.58633
			10	2.69726	2.69577	2.69542	2.65244
			20	2.69502	2.69465	2.69456	2.67359
			100	2.69430	2.69428	2.69428	2.69017

erately high friction losses, the second-order model gives much better results with respect to the accuracy and efficiency in calculations than those obtained by the first-order model.

Case 2.—The system parameters and the valve closure time are: B = 0.5, σ = 0.9, and t_c = 0.0. In these systems with the parameters as shown previously, for example, in a long oil pipeline, a numerical instability

TABLE 4.—Maximum Waterhammer Head Computed

t_c (1)	B (2)	σ (3)	M (4)	Series solution method (5)	Newton-Raphson method (6)	"Algebraic Solution" method (7)	First-order model (8)
5.0	1.0	0.4	4	1.11293	1.11241	1.11231	1.10922
			10	1.11232	1.11224	1.11222	1.11096
			20	1.11223	1.11221	1.11220	1.11157
			100	1.11220	1.11220	1.11220	1.11207
		0.8	8	1.10549	1.10559	1.10561	1.10555
			19	1.10558	1.10559	1.10559	1.10555
			40	1.10558	1.10559	1.10559	1.10557
			100	1.10559	1.10559	1.10559	1.10558
	2.0	0.4	2	1.27440	1.27255	1.27218	1.25901
			5	1.27214	1.27181	1.27174	1.26659
			10	1.27178	1.27169	1.27167	1.26912
			100	1.27165	1.27165	1.27165	1.27140
		0.8	4	1.22586	1.22482	1.22462	1.21844
			10	1.22465	1.22447	1.22443	1.22193
			20	1.22446	1.22441	1.22440	1.22315
			100	1.22440	1.22440	1.22439	1.22414

TABLE 5.—Maximum Waterhammer Head Computed for System with High Friction Losses (Including Error Due to Steady Flow Calculation)

t_c (1)	B (2)	σ (3)	M (4)	Series solution method (5)	Newton-Raphson method (6)	"Algebraic Solution" method (7)	First-order model (8)
0.0	0.5	0.9	5	1.40016	1.32351	1.31713	1.19289
			9	1.35397	1.33539	1.33269	1.26518
			20	1.34896	1.34570	1.34504	1.31517
			90	1.35383	1.35367	1.35364	1.34708

may result. The numbers of reaches are determined from Eq. 35 so that δ becomes 10^{-1}, 10^{-2}, 10^{-3}, and 10^{-4}. In Table 5 the comparison is made. For $M = 5$ and 9 the series solution method gives unstable results due to the error in the steady flow calculations. Under these severe conditions, the Newton-Raphson method gives more stable results than those given by the series solution method. One of the methods to avoid inaccurate calculations is to set steady flow conditions at grid points where the flow should theoretically be steady.

The numerical results, in that case, are shown in Table 6, from which we can see that the computed results are more accurate than those computed by setting the steady flow conditions at time zero.

The results in cases 1, 2 show that the first-order model requires ≈4–5 times the number of reaches required by the second-order model for obtaining a solution with almost the same accuracy. If the calculations are carried out to a given time, $K \cdot \Delta t$, the ratio of the total number of computation points in space and time required by the first-order model to that by the second-order model is

$$\text{ratio: } \frac{(NM+1)}{M+1}\frac{2NMK}{2MK} = \frac{N(NM+1)}{M+1} \quad \text{(40)}$$

in which K = the integer; and N = the ratio of the number of reaches required by the first-order model to that by the second-order model for obtaining a solution with almost the same accuracy. Examination of Eq. 40 shows that the first-order model requires ≈12–25 times the total number of grid points required by the second-order model.

TABLE 6.—Maximum Waterhammer Head Computed for System with High Friction Losses (Not Including Error Due to Steady Flow Calculation)

t_c (1)	B (2)	σ (3)	M (4)	Series solution method (5)	Newton-Raphson method (6)	"Algebraic Solution" method (7)	First-order model (8)
0.0	0.5	0.9	5	1.31261	1.31645	1.31713	1.19289
			9	1.33157	1.33250	1.33269	1.26518
			20	1.34487	1.34501	1.34504	1.31517
			90	1.35363	1.35364	1.35364	1.34708

Summary and Conclusions

How efficiently the second-order model is handled is analysed for the simplified equations governing waterhammer appearance in piping systems. The series solution method and the Newton-Raphson method with the new calculation steps are proposed by the writers through omitting trivial terms computed within the truncation error.

With fewer calculations than required previously, the new methods can yield a solution with the required accuracy without any iteration. Then, compared with the procedure given by the first-order model, the second-order model offers more efficient and accurate procedures, which are applicable not only to systems with high friction losses but also to systems with moderately high friction losses.

However, since the second-order model introduces an error in steady flow calculations, ways to remove or reduce this error are shown.

The validity of the analyses is examined by the numerical computations in which the system parameters are varied widely.

Appendix I.—References

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Appendix II.—Notation

The following symbols are used in this paper:

a^* = wavespeed of pressure pulse;
B = Allievi constant, $a^* V_0/(gH_0)$;
C^+ = advancing characteristic line;
C^- = receding characteristic line;
D = diameter of a pipe;
E = error in the left-hand side of Eq. 13 in iterative process;
F = general expression of coefficient of the friction term;
f = Darcy-Weisbach's friction factor;
g = gravitational acceleration;

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H = head;
H_0 = characteristic head (constant head at an upstream reservoir);
h = dimensionless head, H/H_0;
h_b = dimensionless head at a boundary;
$\tilde{h}_k$ = kth component of the series solution for h;
h^+ = dimensionless head known at point P;
h^- = dimensionless head known at point Q;
K = integer;
L_0 = characteristic length (pipe length);
M = number of reaches;
m = exponent included in the friction term;
N = ratio of the number of reaches;
T = time;
t = dimensionless time, $T/(2L_0/a^*)$;
t_c = dimensionless time for valve closure;
Δt = time step size;
$\tilde{u}_k$ = kth component of the power series of ϵ for v;
V = flow velocity;
V_0 = characteristic velocity (initial steady velocity);
v = dimensionless velocity, V/V_0;
v_b = dimensionless velocity at a boundary;
v_k or $v_k(t_1)$ = kth differential coefficients for v expanded at point P or Q in the Taylor's series;
v^k = kth estimate of velocity v in the Newton-Raphson method;
Δv = correction term;
Δv^k = kth correction term;
v^+ = dimensionless velocity known at point P;
v^- = dimensionless velocity known at point Q;
X = distance coordinate;
x = dimensionless distance coordinate, X/L_0;
Δx = distance step size;
Y = $\sigma/(BM)$;
y = parameter related to time t;
δ = error produced in steady flow calculations;
ϵ = $\delta\Delta t/2^m$; and
σ = $F\,V_0^{m+1}L_0/(gH_0)$.

J. Hydraul. Eng. 1984.110:736-748.