Computers and Mathematics with Applications 57 (2009) 1280–1285

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

# Transient solution of the M/M/c queue with balking and reneging

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ARTICLE INFO

*Article history:*
Received 3 October 2007
Received in revised form 23 November 2008
Accepted 12 January 2009

*Keywords:*
Transient probabilities
M/M/c queue
Time dependent queue
Generating functions
Modified Bessel functions

ABSTRACT

This paper presents an analysis for the M/M/c queue with balking and reneging. It is assumed that arriving customers balk with a fixed probability and renege according to a negative exponential distribution. The generating function technique will be used to obtain the transient solution of system which results in a simple differential equation. Based on the properties of Bessel functions in the solution of this differential equation the probabilities can be extracted in a direct way.



## 1. Introduction

In real life, many queueing situations arise in which there may be tendency of customers to be discouraged by a long queue. As a result, the customers either decide not join the queue (i.e. balk) or depart after joining the queue without getting served due to impatience (i.e. renege). The importance of this system appears in many real life problems such as the situations involving impatient telephone switchboard customers, the hospital emergency rooms handling critical patients, and the inventory systems that store perishable goods [1].

So queueing systems with balking, reneging, or both were studied by many researchers. Haight [2] first presented the M/M/1 queue with balking. Al-Seedy and Kotb [3] considered the transient solution of a single-server system with balking concept. The M/M/1 queue with customers reneging was also proposed in [4]. The combined effects of balking and reneging in the M/M/1/N queue were investigated in [5,6]. The M/M/1/N Queue with balking, reneging and server vacations was analyzed in [7].

On the other hand, multi-server Markovian queues have been widely studied due to a significant role in day-by-day queueing situation. Reynolds [8] gave the stationary solution of a multi-server model with discouragement. A multi-server queueing model with balking and reneging was proposed by using diffusion approximation method (see [9]). Haghighi et al. [10] derived the steady state probabilities for multi-channel M/M/c queue under consideration of balking and reneging concepts. Hillier and Lieberman discussed various aspects of balking and reneging in [11]. Abou El-Ata and Hariri [12] analyzed the multiple servers' queueing system M/M/c/N with balking and reneging. A finite capacity priority queue with discouragement was discussed in [13]. Jain [14] obtained steady state queue size distribution and some other characteristics for M/M/m queue with discouragement and additional servers. The study of M/M/c/N queue with balking, reneging and server breakdowns was presented in [15]. The M/M/c/N queue with balking, reneging and synchronous vacation of partial

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doi:10.1016/j.camwa.2009.01.017

serves was analyzed in [16]. An additional repairman for machine repair problem with balking, and reneging and spares was incorporated in [17]. Recently, Jain and Singh [18] implemented additional servers for M/M/m queueing model with balking and reneging.

In this investigation, the authors will present another technique to compute transient probabilities for the M/M/c queue under consideration of balking and reneging. This technique is a straightforward application of generating functions. The transient probabilities will be expressed in terms of Bessel functions. This is an extension of results obtained for transient probabilities for the multi-server queue given in [19].

This paper is organized as follows. Section 2, gives a description of the queueing model. In Section 3, the equation of system in case balking and reneging is formulated. A simple differential equation is derived and the transient probabilities are obtained by using the properties of Bessel functions in the solution of this differential equation. An equation to evaluate the transient probabilities $P_{c-1}(t)$ is derived in Section 4. Conclusions are given in Section 5.

## 2. System model

In this paper, we deal with the M/M/c queueing system as well as balking and reneging. We consider the following assumptions:

(a) Customers arrive at the system one by one according to a Poisson process with rate $\lambda$. On arrival a customer either decides to join the queue with probability

$$\beta = \text{prob.}\{\text{a unit joins the queue}\}$$

or balk with probability $1-\beta$, where $0 \le \beta < 1$ if $n = c(1)\infty$ and $\beta = 1$ if $n = 0(1)\overline{c-1}$.

(b) After joining the queue, each customer will wait a certain length of time $T$ for service to begin. If it has not begun by then, he will get impatient and leave the queue without getting served. This time $T$ is a random variable with the following density function

$$f(t) = \alpha \exp(-\alpha t), \quad t \ge 0,\ \alpha \ge 0,$$

where $\alpha$ is the rate of time $T$. Since the arrival and the departure of the impatient customers without service are independent, the average reneging rate of the customer can be given by $(n-c)\alpha$. Hence, the used function of customer's average reneging rate is given by

$$r(n) = \begin{cases} 0, & 0 \le n \le c \\ (n-c)\alpha, & n \ge c+1. \end{cases}$$

(c) The customers are served on a first-come, first-served (FCFS) discipline. The service times are assumed to be distributed according to an exponential distribution with the following density function:

$$s(t) = \mu \exp(-\mu t), \quad t \ge 0,\ \mu \ge 0,$$

where $\mu$ is the service rate.

## 3. The transient probabilities for the M/M/c queueing system

In this section, the authors formulate the differential equations for the transient probabilities of the presented queueing system by adding the balking and reneging concepts. In this case, the forward equations for the system

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) + \mu P_1(t), \tag{3.1}$$

$$\frac{dP_n(t)}{dt} = -(\lambda + n\mu)P_n(t) + \lambda P_{n-1}(t) + (n+1)\mu P_{n+1}(t), \quad 1 \le n < c, \tag{3.2}$$

$$\frac{dP_c(t)}{dt} = -(\beta\lambda + c\mu)P_c(t) + \lambda P_{c-1}(t) + (c\mu + \alpha)P_{c+1}(t), \tag{3.3}$$

$$\frac{dP_n(t)}{dt} = -(\beta\lambda + c\mu + (n-c)\alpha)P_n(t) + \beta\lambda P_{n-1}(t) + (c\mu + (n-c+1)\alpha)P_{n+1}(t), \quad n \ge c+1, \tag{3.4}$$

with $P_n(0) = \delta_{ni}$ are obtained, where the function $P_n(t)$ is written for $P_{in}(t)$ which is the probability that there are $n$ customers in the system at time $t$ given that there were $i$ customers initially.

If we define

$$P(z,t) = \sum_{n=0}^{c-1} P_n(t) + \sum_{n=c}^{\infty} P_n(t) z^{n-c+1}, \qquad P(z,0) = z^{\tau(i)} \tag{3.5}$$

with

$$\sum_{n=0}^{c-1} P_n(t) = q_{c-1}(t) \quad \textit{and} \quad \tau(i) = (i+1-c)\left(1 - \sum_{k=0}^{c-1} \delta_{ik}\right), \tag{3.6}$$

the system equations (3.1)–(3.4) yields the following partial differential

$$\frac{\partial P(z,t)}{\partial t}-\alpha(1-z)\frac{\partial P(z,t)}{\partial z}=[(c\mu-\alpha)(z^{-1}-1)+\beta\lambda(z-1)][P(z,t)-q_{c-1}(t)]+\lambda(z-1)P_{c-1}(t) \tag{3.7}$$

The solution of (3.7) is easily obtained as

$$\begin{aligned}P(z,t)&=\exp\{[(c\mu-\alpha)(z^{-1}-1)+\beta\lambda(z-1)]t\}\sum_{\xi=0}^{\infty}\binom{\tau(i)}{\xi}\exp(-\alpha(\tau(i)-\xi)t)z^{\tau(i)-\xi}(1-\exp(-\alpha t))^{\xi}\\&\quad+\int_0^t[\lambda(z-1)P_{c-1}(u)-((c\mu-\alpha)(z^{-1}-1)+\beta\lambda(z-1))q_{c-1}(u)]\\&\quad\times\exp\{[(c\mu-\alpha)(z^{-1}-1)+\beta\lambda(z-1)](t-u)\}\mathrm{d}u.\end{aligned} \tag{3.8}$$

It is well know that if $r=2\sqrt{\beta\lambda(c\mu-\alpha)}$ and $\nu=\sqrt{\beta\lambda/(c\mu-\alpha)}$, then

$$\exp\left(\left(\beta\lambda z+\frac{c\mu-\alpha}{z}\right)t\right)=\sum_{n=-\infty}^{\infty}(\nu z)^n I_n(rt),$$

where $I_n(.)$ is the modified Bessel function. Using this statement in (3.8) and comparing the coefficients of $z^n$ on right and left hand side, we get for $n=1,,2,\ldots$

$$\begin{aligned}\nu^{1-n}P_{n+c-1}(t)&=\sum_{\xi=0}^{\infty}\left[\binom{\tau(i)}{\xi}\exp(-\alpha(\tau(i)-\xi)t)(1-\exp(-\alpha t))^{\xi}\exp(-(\beta\lambda+(c\mu-\alpha))t)\right.\\&\quad\left.\times\ \nu^{1-\tau(i)+\xi}I_{n-\tau(i)+\xi}(rt)\right]+\int_0^t\exp(-(\beta\lambda+(c\mu-\alpha))(t-u))\\&\quad\times[\lambda(I_{n-1}(r(t-u))-\nu I_n(r(t-u)))P_{c-1}(u)+((\beta\lambda+(c\mu-\alpha))\\&\quad\times\nu I_n(r(t-u))-\beta\lambda(I_{n+1}(r(t-u))+I_{n-1}(r(t-u))))q_{c-1}(u)]\mathrm{d}u\end{aligned} \tag{3.9}$$

and for $n=0$

$$\begin{aligned}\nu q_{c-1}(t)&=\sum_{\xi=0}^{\infty}\left[\binom{\tau(i)}{\xi}\exp(-\alpha(\tau(i)-\xi)t)(1-\exp(-\alpha t))^{\xi}\exp(-(\beta\lambda+(c\mu-\alpha))t)\right.\\&\quad\left.\times\ \nu^{1-\tau(i)+\xi}I_{\tau(i)-\xi}(rt)\right]+\int_0^t\exp(-(\beta\lambda+(c\mu-\alpha))(t-u))\\&\quad\times[\lambda(I_1(r(t-u))-\nu I_0(r(t-u)))P_{c-1}(u)+((\beta\lambda+(c\mu-\alpha))\\&\quad\times\nu I_0(r(t-u))-2\beta\lambda(I_1(r(t-u))))q_{c-1}(u)]\mathrm{d}u.\end{aligned} \tag{3.10}$$

As $P(z,t)$ does not contain terms with negative powers of $z$, the right-hand side of (3.9) with $n$ replaced by $-n$, must be zero. Thus,

$$\begin{aligned}&-\sum_{\xi=0}^{\infty}\left[\binom{\tau(i)}{\xi}\exp(-\alpha(\tau(i)-\xi)t)(1-\exp(-\alpha t))^{\xi}\exp(-(\beta\lambda+(c\mu-\alpha))t)\nu^{1-\tau(i)+\xi}I_{n+\tau(i)-\xi}(rt)\right]\\&\quad-\int_0^t\exp(-(\beta\lambda+(c\mu-\alpha))(t-u))[\lambda(I_{n+1}(r(t-u))-\nu I_n(r(t-u)))P_{c-1}(u)]\mathrm{d}u\\&=\int_0^t\exp(-(\beta\lambda+(c\mu-\alpha))(t-u))((\beta\lambda+(c\mu-\alpha))\\&\quad\times\nu I_n(r(t-u))-\beta\lambda(I_{n-1}(r(t-u))+I_{n+1}(r(t-u))))q_{c-1}(u)\mathrm{d}u,\end{aligned} \tag{3.11}$$

where we have used $I_{-k}(.)=I_k(.)$. The usage of (3.11) in (3.9) considerably simplifies the working and results in elegant expression for $P_n(t)$. This yields, for $n=1,2,3,\ldots$

$$\begin{aligned}P_{n+c-1}(t)&=\sum_{\xi=0}^{\infty}\left[\binom{\tau(i)}{\xi}\exp(-\alpha(\tau(i)-\xi)t)(1-\exp(-\alpha t))^{\xi}\exp(-(\beta\lambda+(c\mu-\alpha))t)\right.\\&\quad\left.\times\ \nu^{n-\tau(i)+\xi}(I_{n-\tau(i)+\xi}(rt)-I_{n+\tau(i)-\xi}(rt))\right]+\lambda\nu^{n-1}\int_0^t\exp(-(\beta\lambda+(c\mu\alpha))\\&\quad\times(t-u))(I_{n-1}(r(t-u))-I_{n+1}(r(t-u)))P_{c-1}(u)\mathrm{d}u.\end{aligned} \tag{3.12}$$

## 4. The transient probability $P_{c-1}(t)$

To solve the probabilities $P_n(t)$, $n = 0, 1, 2, \ldots, c-1$, consider the system of Eqs. (3.1) and (3.2) subject to the condition (3.10). The system (3.2) together with (3.1) can be expressed in the form

$$\frac{\mathrm{d}\mathbf{P}(t)}{\mathrm{d}t} = \mathbf{A}\mathbf{P}(t) + (c-1)\mu P_{c-1}(t)\mathbf{e}_{c-1}, \tag{4.1}$$

where

$$\mathbf{P}(t) = (P_0(t), P_1(t), \ldots, P_{c-2}(t))^{\mathrm{T}}, \qquad \mathbf{A} = (a_{kj})_{c-1\times c-1} \tag{4.2}$$

with

$$a_{kj} = \begin{cases} \lambda, & j = k-1, k = 0, 1, 2, \ldots, c-2, \\ -(\lambda + k\mu), & j = k, k = 0, 1, 2, \ldots, c-2, \\ (k+1)\mu, & j = k+1, k = 0, 1, 2, \ldots, c-3, \end{cases}$$

and $\mathbf{e}_{c-1}$ is a column vector of order $c-1$ with the last element 1 and the remaining elements zero.

Taking Laplace transform $\hat{f}(s)$ of $f(t)$, then the solution of (4.1) is given by

$$\hat{\mathbf{P}}(s) = (s\mathbf{I} - \mathbf{A})^{-1}((c-1)\mu\hat{P}_{c-1}(s)\mathbf{e}_{c-1} + \mathbf{P}(0)), \tag{4.3}$$

with

$$\mathbf{P}(0) = (\delta_{i0}, \delta_{i1}, \delta_{i2}, \ldots, \delta_{ic-2})^{\mathrm{T}}. \tag{4.4}$$

Thus, only $\hat{P}_{c-1}(s)$ remains to be found. We observe that if $\mathbf{e} = (1, 1, 1, \ldots, 1)^{\mathrm{T}}$ and $\hat{q}_{c-1}(s) = \hat{\mathbf{e}}\hat{\mathbf{P}}(s) + \hat{P}_{c-1}(s)$, using (3.10) and simplifying, we get

$$\hat{P}_{c-1}(s) = \frac{\sum_{\xi=0}^{\infty}\sum_{g=0}^{\xi}(-1)^g \binom{\tau(i)}{\xi}\binom{\xi}{g}\left(\frac{\Gamma-\sqrt{\Gamma^2-r^2}}{r\ \nu}\right)^{\tau(i)-\xi}\left(\frac{\sqrt{p^2-r^2}}{\sqrt{\Gamma^2-r^2}}\right) - s\mathbf{e}^{\mathrm{T}}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{P}(0)}{s + \beta\lambda - \left(\frac{p-\sqrt{p^2-r^2}}{2}\right) + (c-1)\mu s\mathbf{e}^{\mathrm{T}}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{e}_{c-1}} \tag{4.5}$$

where $p = s + \beta\lambda + (c\mu - \alpha)$ and $\Gamma = p + \alpha(\tau(i) - \xi + g)$.

In (4.3) and (4.5), $(s\mathbf{I} - \mathbf{A})^{-1}$ has to be found. For smaller order matrices the usual procedure can be employed. For higher order matrices, we can follow the procedure given in [20] to get the element of the matrix $(s\mathbf{I} - \mathbf{A})^{-1}$. To this end, let

$$(s\mathbf{I} - \mathbf{A})^{-1} = (\hat{a}_{kj}(s))_{(c-1)\times(c-1)}.$$

We note that $(s\mathbf{I} - \mathbf{A})$ is almost lower triangular. Following [20], we obtain

$$\hat{a}_{kj}(s) = \begin{cases} \frac{1}{(j+1)\mu}\left[\frac{u_{c-1,j+1}(s)\,u_{k,0}(s) - u_{k,j+1}(s)u_{c-1,0}(s)}{u_{c-1,0}(s)}\right], & j = 0, 1, 2, \ldots, c-3 \\ \frac{u_{k,0}(s)}{u_{c-1,0}(s)}, & j = c-2 \end{cases} \tag{4.6}$$

for $k = 0, 1, 2, \ldots, c-2$, where $u_{k,j}(s)$ are recursively given as

$$u_{k,k}(s) = 1, \quad k = 0, 1, 2, \ldots, c-2;$$

$$u_{k+1,k}(s) = \frac{s + \lambda + k\mu}{(k+1)\mu}, \quad k = 0, 1, 2, \ldots, c-3;$$

$$u_{k+1,k-j}(s) = \frac{(s + \lambda + k\mu)u_{k,k-j} - \lambda u_{k-1,k-j}}{(k+1)\mu}, \quad j \le k, k = 1, 2, \ldots, c-3;$$

$$u_{c-1,j}(s) = \begin{cases} [s + \lambda + (c-2)\mu]u_{c-2,j} - \lambda u_{c-3,j}, & j = 0, 1, 2, \ldots, c-3 \\ s + \lambda + (c-2)\mu, & j = c-2 \end{cases}$$

and $u_{k,j}(s) = 0$ for other values of $k$ and $j$.

To facilitate computation, we have suppressed the argument $s$. The advantage in using these relations is that the authors do not evaluate any determinant. Using these in (4.5), we obtain

$$\hat{P}_{c-1}(s) = \frac{\sum_{\xi=0}^{\infty}\sum_{g=0}^{\xi}(-1)^g \binom{\tau(i)}{\xi}\binom{\xi}{g}\left(\frac{\Gamma-\sqrt{\Gamma^2-r^2}}{r\ \nu}\right)^{\tau(i)-\xi}\left(\frac{\sqrt{p^2-r^2}}{\sqrt{\Gamma^2-r^2}}\right) - \sum_{j=0}^{c-2}\sum_{k=0}^{c-2}\delta_{ij}s\hat{a}_{kj}(s)}{s + \beta\lambda - \left(\frac{p-\sqrt{p^2-r^2}}{2}\right) + (c-1)\mu s\sum_{k=0}^{c-2}\hat{a}_{k\,c-2}(s)} \tag{4.7}$$

and for $k = 0, 1, 2, \ldots, c-2$,

$$\hat{P}_k(s) = \sum_{j=0}^{c-2} \delta_{ij}\hat{a}_{kj}(s) + (c-1)\mu\hat{a}_{k\,c-2}(s)\hat{P}_{c-1}(s). \tag{4.8}$$

It is clear that $\hat{a}_{kj}(s)$ are all rational algebraic function in $s$. The cofactor of the $(i, j)th$ element of $(s\mathbf{I} - \mathbf{A})$ is a polynomial of degree $c - 2 - |i - j|$. In particular, the cofactors of the diagonal elements are polynomials in $s$ of degree $c - 2$ with the leading coefficient equal to 1. In fact $u_{c-1,0}(s) = 0$ is the characteristic equation of $\mathbf{A}$. Since $a_{00} = \lambda$, $\lambda \neq 0$, it is also known that the characteristic roots of $\mathbf{A}$ are all distinct and negative [21]. Hence the inverse transform $\tilde{a}_{kj}(t)$ of $\hat{a}_{kj}(s)$ can be obtained by partial fraction decomposition.

Let $s_k$, $k = 0, 1, 2, \ldots, c-2$ be the characteristic roots of matrix $\mathbf{A}$. Then

$$\hat{a}_{kj}(s) = \sum_{m=0}^{c-2} \frac{\rho_{kj}^{(m)}}{s - s_m}, \tag{4.9}$$

where the constants $\rho_{kj}^{(m)}$ are given by

$$\rho_{kj}^{(m)} = \lim_{s\to s_m} ((s - s_m)\hat{a}_{kj}(s)). \tag{4.10}$$

Thus, we can write $\tilde{a}_{kj}(t)$ in the following form:

$$\tilde{a}_{kj}(t) = \sum_{m=0}^{c-2} \rho_{kj}^{(m)} \exp(s_m t). \tag{4.11}$$

Similarly,

$$\sum_{k=0}^{c-2} (c-1)s\hat{a}_{kj}(s) = c - 1 + \hat{b}_j(s), \quad j = 0, 1, 2, \ldots, c-2, \tag{4.12}$$

where $\hat{b}_j(s)$ can be resolved as

$$\hat{b}_j(s) = \sum_{m=0}^{c-2} \frac{B_j^{(m)}}{s - s_m} \tag{4.13}$$

with

$$B_j^{(m)} = \lim_{s\to s_m} ((s - s_m)(c-1)\sum_{k=0}^{c-2} s\hat{a}_{kj}(s)). \tag{4.14}$$

Then the inverse transform $b_j(t)$ of $\hat{b}_j(s)$ is given by

$$b_j(t) = \sum_{m=0}^{c-2} B_j^{(m)} \exp(s_m t), \quad j = 0, 1, 2, \ldots, c-2. \tag{4.15}$$

Using (4.12) in (4.7), then

$$\hat{P}_{c-1}(s) = \frac{\sum_{\xi=0}^{\infty}\sum_{g=0}^{\xi}(-1)^g \binom{\tau(i)}{\xi}\binom{\xi}{g}\left(\frac{\Gamma-\sqrt{\Gamma^2-r^2}}{r\,\nu}\right)^{\tau(i)-\xi}\sqrt{\frac{p^2-r^2}{\Gamma^2-r^2}} - \sum_{j=0}^{c-2}\delta_{ij}\left(\frac{c-1+\hat{b}_j(s)}{c-1}\right)}{\left(\frac{p+\sqrt{p^2-r^2}}{2}\right) + \alpha - \mu + \mu\hat{b}_{c-2}(s)}. \tag{4.16}$$

Hence (4.16) simplifies to

$$\begin{aligned}\hat{P}_{c-1}(s) = {}& \left(\frac{2}{r}\right)\left(\frac{p-\sqrt{p^2-r^2}}{r}\right) \\ & \times \left[\sum_{\xi=0}^{\infty}\sum_{g=0}^{\xi}(-1)^g \binom{\tau(i)}{\xi}\binom{\xi}{g}\left(\frac{\Gamma-\sqrt{\Gamma^2-r^2}}{r\,\nu}\right)^{\tau(i)-\xi}\sqrt{\frac{p^2-r^2}{\Gamma^2-r^2}} - \sum_{j=0}^{c-2}\delta_{ij}\left(\frac{c-1+\hat{b}_j(s)}{c-1}\right)\right] \\ & \times \left[1 - \frac{1}{\sqrt{\beta\lambda(c\mu-\alpha)}}\left(\frac{p-\sqrt{p^2-r^2}}{r}\right)\left(\mu - \alpha - \mu\hat{b}_{c-2}(s)\right)\right]^{-1}\end{aligned} \tag{4.17}$$

which on inversion yields

$$
\begin{aligned}
P_{c-1}(t) = & \sum_{m=0}^{\infty}\left(\frac{2}{r}\right)^{m+1}\sum_{k=0}^{m}(-1)^k\binom{m}{k}(\mu-\alpha)^{m-k}\mu^k\int_0^t b_{c-2}^{*k}(t-u)\left\{\int_0^u\sum_{\xi=0}^{\infty}\sum_{g=0}^{\xi}(-1)^g\right. \\
& \times\binom{\tau(i)}{\xi}\binom{\xi}{g}\nu^{\xi-\tau(i)}\exp(-(\beta\lambda+(c\mu-\alpha)+\alpha\{\tau(i)-\xi+g\})(u-\theta))I_{\tau(i)-\xi}(r(u-\theta)) \\
& \times(m+1)\left[\int_0^{\theta}\left(\delta(\theta-y)+\frac{I_1(r(\theta-y))}{r(\theta-y)}\right)\exp(-(\beta\lambda+(c\mu-\alpha))\,y)\left(\frac{I_{m+1}(ry)}{y}\right)\mathrm{d}y\right]\mathrm{d}\theta \\
& \left.-\ (m+1)\sum_{j=0}^{c-2}\delta_{ij}\int_0^u\left(\delta(u-x)+\left(\frac{b_j(u-x)}{c-1}\right)\right)\exp(-(\beta\lambda+(c\mu-\alpha))x)\left(\frac{I_{m+1}(rx)}{x}\right)\mathrm{d}x\right\}\mathrm{d}u \qquad (4.18)
\end{aligned}
$$

where $b_{c-2}^{*k}(t)$ is k-fold convolution of $b_{c-2}(t)$ with itself. We note that $b_{c-2}^{*0}(t)=\delta(t)$.

Finally, for $k=0,1,2,\ldots,c-2$,

$$
P_k(t)=\sum_{j=0}^{c-2}\delta_{ij}a_{kj}(t)+(c-1)\mu\int_0^t a_{k\,c-2}(u)P_{c-1}(t-u)\mathrm{d}u. \qquad (4.19)
$$

Thus (3.12), (4.18) and (4.19) completely determine all the state probabilities of the queue size.

## 5. Conclusion

In this paper, the M/M/c queue with balking and reneging was considered. Equations for system in the cases of balking and reneging were formulated. The authors also gave the transient probabilities of the queue size, by using the generating function technique and the properties of Bessel functions.

## Acknowledgments

The authors would like to thank Prof. P. R. Parthasarathy, Indian Institute of Technology Madras, India, for his constructive comments and the referees for their valuable suggestions that led to considerable improvement in the presentation of this paper.

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