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An approach to group decision making with heterogeneous incomplete uncertain preference relations ☆

Zhen Zhang *, Chonghui Guo [1]

Institute of Systems Engineering, Dalian University of Technology, No. 2 Linggong Road, Ganjingzi, 116024 Dalian, Liaoning, PR China

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ABSTRACT

For practical group decision making problems, decision makers tend to provide heterogeneous uncertain preference relations due to the uncertainty of the decision environment and the difference of cultures and education backgrounds. Sometimes, decision makers may not have an in-depth knowledge of the problem to be solved and provide incomplete preference relations. In this paper, we focus on group decision making (GDM) problems with heterogeneous incomplete uncertain preference relations, including uncertain multiplicative preference relations, uncertain fuzzy preference relations, uncertain linguistic preference relations and intuitionistic fuzzy preference relations. To deal with such GDM problems, a decision analysis method is proposed. Based on the multiplicative consistency of uncertain preference relations, a bi-objective optimization model which aims to maximize both the group consensus and the individual consistency of each decision maker is established. By solving the optimization model, the priority weights of alternatives can be obtained. Finally, some illustrative examples are used to show the feasibility and effectiveness of the proposed method.

1. Introduction

The increasing complexity of the socio-economic environment makes it less and less possible for a single decision maker (DM) to consider all relevant aspects of a decision making problem (Kim, Choi, & Kim, 1999). Therefore, many decision making problems in the real world are usually conducted by decision groups, and group decision making (GDM) problem has long been identified as a hot topic in decision science research area (Hwang & Lin, 1987).

For a typical GDM problem, decision makers are usually asked to provide their preferences over a set of alternatives (criteria). As an effective tool, preference relation has been widely used to express decision makers' preference information through pairwise comparisons. Up to now, many formats of preference relations have been developed (Xu, 2007b), such as multiplicative preference relation (Herrera, Herrera-Viedma, & Chiclana, 2001; Saaty, 1980), fuzzy preference relation (Herrera-Viedma, Chiclana, Herrera, & Alonso, 2007; Tanino, 1984) and linguistic preference relation (Herrera, Herrera-Viedma, & Verdegay, 1996; Xu, 2006; Xu, 2008). But due to the uncertainty of decision environment and the lack of decision makers' knowledge, preference relations given by decision makers sometimes are uncertain ones (Liu, Zhang, & Wang, 2012; Xu, 2004b). As a result, many publications have focused on deriving priority weights from uncertain preference relations (Chen & Zhou, 2012; Gong, Li, Zhou, & Yao, 2009; Wang, Yang, & Xu, 2005; Wu, Li, Li, & Duan, 2009; Xu & Chen, 2008a).

For some complex GDM problems defined with high uncertainty, decision makers may be of different culture and education background and may have different levels of knowledge about the decision making problems (Herrera-Viedma, Herrera, & Chiclana, 2002; Palomares, Rodríguez, & Martínez, 2013). On the other hand, decision makers sometimes are distributed in different areas and it may be difficult for them to reach an agreement on which type of preference relations can be used. In such situations, decision makers may tend to express their preference using different formats of preference relations according to their own will. In recent years, group decision making with heterogeneous preference information has received more and more attention (Delgado, Herrera, Herrera-Viedma, & Martínez, 1998; Espinilla, Palomares, Martínez, & Ruan, 2012; Fan, Xiao, & Hu, 2004; Li, Huang, & Chen, 2010; Pérez, Alonso, Cabrerizo, Lu, & Herrera-Viedma, 2011). For instance, Herrera-Viedma et al. (2002) presented a consensus model for multi-person decision making problems with different

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* Corresponding author. Tel.: +86 411 84708007; fax: +86 411 84707425.
E-mail addresses: zhangzhen@mail.dlut.edu.cn (Z. Zhang), guochonghui@tsinghua.org.cn (C. Guo).
[1] Tel.: +86 411 84708007; fax: +86 411 84707425.

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preference structures to help experts change their opinions and obtain a degree of consensus. Herrera, Martínez, and Sánchez (2005) developed an aggregation process to combine different types of preference relations, such as linguistic, numerical and interval-valued information. Fan, Ma, Jiang, Sun, and Ma (2006) established a goal programming model to solve group decision making problems where the preference information on alternatives is represented in multiplicative preference relations and fuzzy preference relations. Wang and Fan (2007) investigated the aggregation of fuzzy preference relations and multiplicative preference relations. In their approach, they presented two optimization aggregation approaches to determine the relative weights of individual fuzzy preference relations so that they can be aggregated into a collective fuzzy preference relation. Dong, Xu, and Yu (2009) proposed a linguistic multi-person decision making model based on linguistic preference relations which can integrate fuzzy preference relations, different types of multiplicative preference relations and multi-granular linguistic preference relations. In order to deal with GDM problems with heterogeneous incomplete preference relations, including multiplicative preference relations, fuzzy preference relations and linguistic preference relations, Fan and Zhang (2010) established a goal programming model to derive the collective evaluation of alternatives. Like Fan and Zhang (2010)'s study, Xu (2011) considered four formats of incomplete preference relations and established a quadratic programming model to obtain the ranking of alternatives. Pérez, Cabrerizo, and Herrera-Viedma (2010) presented a mobile decision support system for dynamic group decision making with fuzzy preference relations, orderings, utility functions and multiplicative preference relations, in which mobile technologies are applied and the set of alternatives can change throughout the process. Pérez, Cabrerizo, and Herrera-Viedma (2011b) also developed a mobile GDM model for changeable decision environments which allows decision makers to express their preferences using heterogeneous preference relations, including fuzzy preference relations and multi-granularity linguistic preference relations. In a recent work, Palomares et al. (2013) proposed a consensus model in which decision makers can express their opinions by using different types of information, capable of dealing with large groups of decision makers, which incorporates the management of the group's attitude towards consensus by means of the proposed Attitude-OWA operator.

From the above analysis, a lot of studies have been conducted to deal with GDM with heterogeneous preference relations and previous studies have significantly advanced the field of GDM. However, most of the research focuses on GDM problems with certain preference relations. There is very little literature addressing GDM problems with heterogeneous uncertain preference relations. On the other hand, for actual GDM problems there may be cases in which decision makers do not have an in-depth knowledge of the problem to be solved. In such cases, decision makers may not put their opinions forward about certain aspects of the problem, and as a result incomplete preference relations may be obtained (Alonso, Herrera-Viedma, Chiclana, & Herrera, 2009; Alonso, Herrera-Viedma, Chiclana, & Herrera, 2010; Herrera-Viedma et al., 2007; Zhang & Guo, 2013). Considering such situations, the main contribution of this paper is to propose a GDM approach to deriving priority weights from heterogeneous incomplete uncertain preference relations, including uncertain multiplicative preference relations, uncertain fuzzy preference relations, uncertain linguistic preference relations and intuitionistic fuzzy preference relations, which can allow decision makers to express their preference information over alternatives more flexibly. For this purpose, this paper first defines the group consensus index and the collective individual consistency index for the four types of incomplete uncertain preference relations under group decision making environment. Afterwards, a bi-objective optimization model, which aims to obtain both the maximum group consensus and collective individual consistency, is proposed to derive the priority weights.

To do so, the rest of this paper is organized as follows. Section 2 presents some concepts and preliminaries related to incomplete uncertain preference relations. In Section 3, we give a description of the group decision making problem with heterogeneous incomplete uncertain preference relations. Section 4 proposed a bi-objective optimization model to address the group decision making problem. In Section 5, we give some illustrative examples to show the feasibility and effectiveness of the proposed method. Section 6 gives a discussion on the advantages and limitations about the proposed approach. Finally, we conclude this paper in Section 7.

2. Preliminaries

In this section, we present some basic concepts and preliminaries related to incomplete uncertain preference relations, including uncertain multiplicative preference relations, uncertain fuzzy preference relations, uncertain linguistic preference relations and intuitionistic fuzzy preference relations.

For the convenience of analysis, let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set of alternatives, where x_i denotes the ith alternative, $i \in \{1, 2, \ldots, n\} = N$. In addition, we denote the priority weight vector obtained from a preference relation by $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$, such that $\sum_{i=1}^{n} w_i = 1,\ w_i \geqslant 0,\ i \in N$.

Definition 1 (*Saaty and Vargas, 1987*). A matrix $\tilde{A} = (\tilde{a}_{ij})_{n\times n}$ is called an uncertain multiplicative preference relation if $\tilde{a}_{ij}$ satisfies $\tilde{a}_{ij} = [a_{ij}^-, a_{ij}^+],\ a_{ij}^+ \geqslant a_{ij}^-,\ a_{ij}^- a_{ji}^+ = a_{ij}^+ a_{ji}^- = 1,\ a_{ii}^+ = a_{ii}^- = 1$, where $\tilde{a}_{ij}$ is the interval-valued preference degree to which the alternative x_i is preferred to x_j, and $a_{ij}^-,\ a_{ij}^+ \in \{1/9, 1/8, 1/7, \ldots, 1/2, 1, 2, \ldots, 7, 8, 9\}$, $i,\ j \in N$.

Definition 2 (Wang et al., 2005). Let $\tilde{A} = (\tilde{a}_{ij})_{n\times n} = ([a_{ij}^-, a_{ij}^+])_{n\times n}$ be an interval multiplicative preference relation. If there exists a positive vector $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ such that the following convex feasible region

$$\Theta = \left\{ w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}} | a_{ij}^- \leqslant \frac{w_i}{w_j} \leqslant a_{ij}^+,\ w_i > 0, i,j \in N, \sum_{i=1}^{n} w_i = 1 \right\} \tag{2.1}$$

is nonempty, then $\tilde{A}$ is called a consistent interval multiplicative preference relation.

Definition 3 (Xu, 2004b). A matrix $\tilde{B} = (\tilde{b}_{ij})_{n\times n}$ is called an uncertain fuzzy preference relation if $\tilde{b}_{ij}$ satisfies $\tilde{b}_{ij} = [b_{ij}^-, b_{ij}^+]$, $b_{ij}^+ \geqslant b_{ij}^-,\ b_{ij}^- + b_{ji}^+ = b_{ij}^+ + b_{ji}^- = 1,\ b_{ii}^+ = b_{ii}^- = 0.5$, where $\tilde{b}_{ij}$ is the interval-valued preference degree to which the alternative x_i is preferred to x_j, and $b_{ij}^-,\ b_{ij}^+ \in [0, 1],\ i,\ j \in N$.

Definition 4 (Xu and Chen, 2008a). Let $\tilde{B} = (\tilde{b}_{ij})_{n\times n} = ([b_{ij}^-, b_{ij}^+])_{n\times n}$ be an interval fuzzy preference relation. If there exists a positive vector $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ such that the following convex feasible region

$$\Theta = \left\{ w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}} | b_{ij}^- \leqslant \frac{w_i}{w_i + w_j} \leqslant b_{ij}^+, w_i > 0, i,j \in N, \sum_{i=1}^{n} w_i = 1 \right\}. \tag{2.2}$$

is nonempty, then $\tilde{B}$ is called a consistent interval fuzzy preference relation.

Definition 5 (Xu, 2007b). Let $S=\{s_\alpha|\alpha=0,1,\ldots,T\}$ be a linguistic term set with odd cardinality as defined by Herrera et al. (1996) and $I(s_\alpha)$ denote the label index of the linguistic term s_α, i.e. $I(s_\alpha)=\alpha$, then a matrix $\tilde{L}=(\tilde{l}_{ij})_{n\times n}$ is called an uncertain linguistic preference relation if $\tilde{l}_{ij}$ satisfies $\tilde{l}_{ij}=[l_{ij}^-,l_{ij}^+]$, $I(l_{ij}^+)\geqslant I(l_{ij}^-)$, $I(l_{ij}^-)+I(l_{ji}^+)=I(l_{ij}^+)+I(l_{ji}^-)=T$, $l_{ii}^+=l_{ii}^-=s_{T/2}$, where $\tilde{l}_{ij}$ is the interval-valued linguistic preference degree to which the alternative x_i is preferred to x_j, and $l_{ij}^-,\ l_{ij}^+\in S,\ i,\ j\in N$.

Gao and Peng (2011) proposed a formula to transform an uncertain linguistic preference relation $\tilde{L}=(\tilde{l}_{ij})_{n\times n}=([l_{ij}^-,l_{ij}^+])_{n\times n}$ into an uncertain fuzzy preference relation $\tilde{B}=(\tilde{b}_{ij})_{n\times n}=([b_{ij}^-,b_{ij}^+])_{n\times n}$ as follows:

$$\tilde{b}_{ij}=[b_{ij}^-,b_{ij}^+]=\left[\frac{I(l_{ij}^-)}{T},\frac{I(l_{ij}^+)}{T}\right],\quad i,\ j\in N, \tag{2.3}$$

where $T+1$ is the cardinality of the linguistic term set as defined in Definition 5.

Definition 6 (Xu, 2007a). An intuitionistic fuzzy preference relation $\tilde{R}$ is represented by a matrix $\tilde{R}=(\tilde{r}_{ij})_{n\times n}$ with $\tilde{r}_{ij}=\langle(x_i,x_j),\ u(x_i,x_j),\ v(x_i,x_j)\rangle,\ i,\ j\in N$. For convenience, let $\tilde{r}_{ij}=(u_{ij},v_{ij}),\ i,\ j\in N$, where $\tilde{r}_{ij}$ is an intuitionistic fuzzy value consisting of the certainty degree u_{ij} to which x_i is preferred to x_j and the certainty degree v_{ij} to which x_i is nonpreferred to x_j, and $u_{ij},\ v_{ij}$ satisfy $0\leqslant u_{ij}+v_{ij}\leqslant 1,\quad u_{ji}=v_{ij},\ v_{ji}=u_{ij},\ u_{ii}=v_{ii}=0.5$, and $\pi_{ij}=1-u_{ij}-v_{ij}$ is interpreted as the uncertainty degree to which x_i is preferred to $x_j,\ i,\ j\in N$.

Definition 7 (Xu, 2007a). Let $\tilde{R}=(\tilde{r}_{ij})_{n\times n}=((u_{ij},v_{ij}))_{n\times n}$ be an intuitionistic fuzzy preference relation. If there exists a positive vector $w=(w_1,w_2,\ldots,w_n)^{\mathrm{T}}$ such that the following convex feasible region

$$\Theta=\left\{w=(w_1,w_2,\ldots,w_n)^{\mathrm{T}}|u_{ij}\leqslant\frac{w_i}{w_i+w_j}\leqslant 1-v_{ij},w_i>0,i,j\in N,\sum_{i=1}^{n}w_i=1\right\}. \tag{2.4}$$

is nonempty, then $\tilde{R}$ is called a consistent intuitionistic fuzzy preference relation.

According to Xu (2006), a preference relation is called an incomplete preference relation if some of the elements of a preference relation cannot be given by decision makers, and other elements can be provided. If all the unknown elements can be obtained by other known elements, the preference relation is considered acceptable. Based on the theorem given by Xu (2006), if an incomplete uncertain preference relation is acceptable, then there exists at least one known element (except diagonal elements) in each line or each column for the preference relation, i.e. at least $n-1$ judgments for the alternatives should be provided. In this paper, all the incomplete preference relations concerned are acceptable ones.

3. Description of the group decision making problem

As denoted in Section 2, let $X=\{x_1,x_2,\ldots,x_n\}$ be a finite set of alternatives, where x_i denotes the ith alternative, $i\in\{1,2,\ldots,n\}=N$. In addition, let $D=\{d_1,d_2,\ldots,d_q\}$ denote the set of decision makers, where d_k denotes the kth decision maker, $k\in\{1,2,\ldots,q\}=K$, and we denote the weight vector of the decision makers by $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_q)^{\mathrm{T}}$, such that $\sum_{k=1}^{q}\lambda_k=1,\ 0\leqslant\lambda_k\leqslant 1,\ k\in K$. Here decision makers provide their preferences over the alternatives through pairwise comparisons, i.e. the preference information is given as preference relations. The preference relations provided by them are the four formats of acceptable incomplete uncertain preference relations as described in Section 2.

According to the diversity of the preference relations, the set of decision makers can be divided into four subsets, i.e. $D=\{D_1,D_2,D_3,D_4\}$. Specifically, let $D_1=\{d_1,d_2,\ldots,d_{h_1}\}$, $D_2=\{d_{h_1+1},d_{h_1+2},\ldots,d_{h_2}\}$, $D_3=\{d_{h_2+1},d_{h_2+2},\ldots,d_{h_3}\}$, $D_4=\{d_{h_3+1},d_{h_3+2},\ldots,d_q\}$. For $d_k\in D_1$, the incomplete uncertain multiplicative preference relation is denoted as $\tilde{A}^k=(\tilde{a}_{ij}^k)_{n\times n}=([a_{ij}^{k-},a_{ij}^{k+}])_{n\times n}$, $k\in\{1,2,\ldots,h_1\}=K_1$; for $d_k\in D_2$, the incomplete uncertain fuzzy preference relation is denoted as $\tilde{B}^k=(\tilde{b}_{ij}^k)_{n\times n}=([b_{ij}^{k-},b_{ij}^{k+}])_{n\times n}$, $k\in\{h_1+1,h_1+2,\ldots,h_2\}=K_2$; for $d_k\in D_3$, the incomplete uncertain linguistic preference relation is denoted as $\tilde{L}^k=(\tilde{l}_{ij}^k)_{n\times n}=([l_{ij}^{k-},l_{ij}^{k+}])_{n\times n}$, $k\in\{h_2+1,h_2+2,\ldots,h_3\}=K_3$; for $d_k\in D_4$, the incomplete intuitionistic fuzzy preference relation is denoted as $\tilde{R}^k=(\tilde{r}_{ij}^k)_{n\times n}=\left((u_{ij}^k,v_{ij}^k)\right)_{n\times n}$, $k\in\{h_3+1,h_3+2,\ldots,q\}=K_4$. The missing elements of the uncertain preference relations are denoted by φ. For the convenience of analysis, we also introduce indication matrices for these four formats of incomplete uncertain preference relations to indicate whether the elements of the preference relations are known or not (Xu, 2004a). The indication matrix $\Delta^k=(\delta_{ij}^k)_{n\times n}$ for the kth incomplete uncertain preference relation can be defined as

$$\delta_{ij}^k=\begin{cases}1 & \text{for } \tilde{a}_{ij}^k\neq\varphi\\ 0 & \text{for } \tilde{a}_{ij}^k=\varphi\end{cases},\quad i,\ j\in N,\ k\in K_1, \tag{3.1a}$$

$$\delta_{ij}^k=\begin{cases}1 & \text{for } \tilde{b}_{ij}^k\neq\varphi\\ 0 & \text{for } \tilde{b}_{ij}^k=\varphi\end{cases},\quad i,\ j\in N,\ k\in K_2, \tag{3.1b}$$

$$\delta_{ij}^k=\begin{cases}1 & \text{for } \tilde{l}_{ij}^k\neq\varphi\\ 0 & \text{for } \tilde{l}_{ij}^k=\varphi\end{cases},\quad i,\ j\in N,\ k\in K_3, \tag{3.1c}$$

$$\delta_{ij}^k=\begin{cases}1 & \text{for } \tilde{r}_{ij}^k\neq\varphi\\ 0 & \text{for } \tilde{r}_{ij}^k=\varphi\end{cases},\quad i,\ j\in N,\ k\in K_4. \tag{3.1d}$$

The group decision making problem to be solved in this paper is to obtain the priority weight vector for the n alternatives based on the incomplete uncertain preference relations provided by the decision makers so that the alternatives can be ranked.

4. The proposed method for GDM

In this section, we present an approach to dealing with the group decision making problem. The basic ideas of the proposed approach are as follows. First, for a group decision making problem, the opinion from each decision maker should be as close to those from other decision makers as possible, i.e. the group consensus should be considered. Second, for each individual preference relation, the consistency should also be considered. With the two points in mind, an approach to group decision making with heterogeneous incomplete uncertain preference relations is proposed.

The resolution procedure of the method is as follows. First, decision makers are invited to express their preference over the alternatives using different formats of preference relations according to their own will, and then the indication matrices of the preference relations can be obtained by Eqs. (3.1a), (3.1b), (3.1c) and (3.1d). Afterwards, a bi-objective optimization model which aims to maximize the group consensus and the individual consistency of each decision maker is established. The priority weight of each alternative is then calculated by solving the bi-objective optimization model using Zimmermann's max–min approach (Zimmermann, 1978). Based on the priority weights, the alternatives can be compared and ranked. Finally, the optimal alternative can be selected. In the rest of this section, we will illustrate how to establish the bi-objective optimization model. For convenience, the notations defined in Section 3 are utilized throughout this section, and let $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ be the priority weight vector obtained from the group's judgments such that $\sum_{i=1}^{n} w_i = 1,\ w_i > 0,\ i \in N$.

4.1. Calculation of the group consensus index

In this subsection, we calculate the group consensus index. Before defining the index, we first calculate the individual priority weight intervals based on each individual preference relation.

4.1.1. Uncertain multiplicative preference relations

By Definition 2, if the uncertain multiplicative preference relation provided by the kth decision maker $(k \in K_1)$ is consistent with the group's opinion, one has

$$a_{ij}^{k-} \leqslant \frac{w_i}{w_j} \leqslant a_{ij}^{k+},\ \text{for all } \delta_{ij}^k = 1,\ i, j \in N,\ i \neq j,\ k \in K_1. \tag{4.1}$$

Obviously, Eq. (4.1) can be written as

$$a_{ij}^{k-} w_j \leqslant w_i \leqslant a_{ij}^{k+} w_j,\ \text{for all } \delta_{ij}^k = 1,\ i, j \in N,\ i \neq j,\ k \in K_1, \tag{4.2}$$

i.e.

$$\delta_{ij}^k a_{ij}^{k-} w_j \leqslant \delta_{ij}^k w_i \leqslant \delta_{ij}^k a_{ij}^{k+} w_j,\ i, j \in N,\ i \neq j,\ k \in K_1. \tag{4.3}$$

Summing all the entries of Eq. (4.3) for all $j \in N,\ j \neq i$, we have

$$\sum_{\substack{j=1\\ j\neq i}}^{n} \delta_{ij}^k a_{ij}^{k-} w_j \leqslant \sum_{\substack{j=1\\ j\neq i}}^{n} \delta_{ij}^k w_i \leqslant \sum_{\substack{j=1\\ j\neq i}}^{n} \delta_{ij}^k a_{ij}^{k+} w_j,\ i \in N,\ k \in K_1. \tag{4.4}$$

As $\sum_{\substack{j=1\\ j\neq i}}^{n} \delta_{ij}^k \neq 0$, it follows that

$$\frac{1}{\sum_{\substack{j=1\\ j\neq i}}^{n} \delta_{ij}^k} \sum_{\substack{j=1\\ j\neq i}}^{n} \delta_{ij}^k a_{ij}^{k-} w_j \leqslant w_i \leqslant \frac{1}{\sum_{\substack{j=1\\ j\neq i}}^{n} \delta_{ij}^k} \sum_{\substack{j=1\\ j\neq i}}^{n} \delta_{ij}^k a_{ij}^{k+} w_j,\ i \in N. \tag{4.5}$$

4.1.2. Uncertain fuzzy preference relations

By Definition 4, if the uncertain fuzzy preference relation provided by the kth decision maker $(k \in K_2)$ is consistent with the group's opinion, the following inequality should hold:

$$b_{ij}^{k-} \leqslant \frac{w_i}{w_i + w_j} \leqslant b_{ij}^{k+},\ \text{for all } \delta_{ij}^k = 1,\ i, j \in N,\ i \neq j,\ k \in K_2, \tag{4.6}$$

namely

$$\frac{b_{ij}^{k-}}{1 - b_{ij}^{k-}} w_j \leqslant w_i \leqslant \frac{b_{ij}^{k+}}{1 - b_{ij}^{k+}} w_j,\ \text{for all } \delta_{ij}^k = 1,\ i, j \in N,\ i \neq j,\ k \in K_2. \tag{4.7}$$

As $b_{ij}^{k-} + b_{ji}^{k+} = 1$ and $b_{ij}^{k+} + b_{ji}^{k-} = 1$, we have

$$\frac{b_{ij}^{k-}}{b_{ji}^{k+}} w_j \leqslant w_i \leqslant \frac{b_{ij}^{k+}}{b_{ji}^{k-}} w_j,\ \text{for all } \delta_{ij}^k = 1,\ i, j \in N,\ i \neq j,\ k \in K_2, \tag{4.8}$$

i.e.

$$\delta_{ij}^k \frac{b_{ij}^{k-}}{b_{ji}^{k+}} w_j \leqslant \delta_{ij}^k w_i \leqslant \delta_{ij}^k \frac{b_{ij}^{k+}}{b_{ji}^{k-}} w_j,\ i, j \in N,\ i \neq j,\ k \in K_2. \tag{4.9}$$

Summing all the entries of Eq. (4.9) for all $j \in N,\ j \neq i$, we have

$$\sum_{\substack{j=1\\ j\neq i}}^{n} \delta_{ij}^k \frac{b_{ij}^{k-}}{b_{ji}^{k+}} w_j \leqslant \sum_{\substack{j=1\\ j\neq i}}^{n} \delta_{ij}^k w_i \leqslant \sum_{\substack{j=1\\ j\neq i}}^{n} \delta_{ij}^k \frac{b_{ij}^{k+}}{b_{ji}^{k-}} w_j,\ i \in N,\ k \in K_2. \tag{4.10}$$

As $\sum_{\substack{j=1\\ j\neq i}}^{n} \delta_{ij}^k \neq 0$, one has

$$\frac{1}{\sum_{\substack{j=1\\ j\neq i}}^{n} \delta_{ij}^k} \sum_{\substack{j=1\\ j\neq i}}^{n} \delta_{ij}^k \frac{b_{ij}^{k-}}{b_{ji}^{k+}} w_j \leqslant w_i \leqslant \frac{1}{\sum_{\substack{j=1\\ j\neq i}}^{n} \delta_{ij}^k} \sum_{\substack{j=1\\ j\neq i}}^{n} \delta_{ij}^k \frac{b_{ij}^{k+}}{b_{ji}^{k-}} w_j,\ i \in N. \tag{4.11}$$

4.1.3. Uncertain linguistic preference relation

As the elements of linguistic preference relations are linguistic terms, a priority weight vector cannot be directly derived from a linguistic preference relation. In order to integrate linguistic preference relations with other numerical preference relations and output a group priority weight vector, linguistic preference relations usually need to be transformed into other types of numerical preference relations. For instance, in Fan and Zhang (2010), the linguistic preference relations are transformed into fuzzy preference relations first and the final priority weight vector is derived based on the multiplicative consistency of fuzzy preference relations. In this paper, we also adopt similar strategy, i.e. transforming uncertain linguistic preference relations into uncertain fuzzy preference relations and utilizing the multiplicative consistency of uncertain fuzzy preference relations.

By Eq. (2.3), we can transform each uncertain linguistic preference relation $\tilde{L}^k,\ k \in K_3$ into an uncertain fuzzy preference relation $\tilde{B}^k = ([b_{ij}^{k-}, b_{ij}^{k+}])_{n\times n},\ k \in K_3$ as

$$\left[b_{ij}^{k-}, b_{ij}^{k+}\right] = \begin{cases} \left[\frac{I\left(l_{ij}^{k-}\right)}{T^k}, \frac{I\left(l_{ij}^{k+}\right)}{T^k}\right] & \delta_{ij}^k = 1, \\ \varphi & \delta_{ij}^k = 0. \end{cases},\ i, j \in N,\ k \in K_3, \tag{4.12}$$

where $T^k + 1$ is the cardinality of the linguistic term set used by the kth decision maker, $k \in K_3$.

Afterwards, the multiplicative consistency of uncertain fuzzy preference relations can be utilized. By Eqs. (4.8) and (4.11), the priority weights obtained from the kth decision maker's uncertain linguistic preference relation should satisfy

$$\frac{I\left(l_{ij}^{k-}\right)}{T^k} \Bigg/ \frac{I\left(l_{ji}^{k+}\right)}{T^k} w_j \leqslant w_i \leqslant \frac{I\left(l_{ij}^{k+}\right)}{T^k} \Bigg/ \frac{I\left(l_{ji}^{k-}\right)}{T^k} w_j,$$
$$\text{for all } \delta_{ij}^k = 1,\ i, j \in N,\ i \neq j,\ k \in K_3, \tag{4.13}$$

or

$$\frac{1}{\sum_{\substack{j=1\\ j\neq i}}^{n} \delta_{ij}^k} \sum_{\substack{j=1\\ j\neq i}}^{n} \delta_{ij}^k \frac{I\left(l_{ij}^{k-}\right)}{T^k} \Bigg/ \frac{I\left(l_{ji}^{k+}\right)}{T^k} w_j \leqslant w_i$$
$$\leqslant \frac{1}{\sum_{\substack{j=1\\ j\neq i}}^{n} \delta_{ij}^k} \sum_{\substack{j=1\\ j\neq i}}^{n} \delta_{ij}^k \frac{I\left(l_{ij}^{k+}\right)}{T^k} \Bigg/ \frac{I\left(l_{ji}^{k-}\right)}{T^k} w_j,\ i \in N, \tag{4.14}$$

i.e.

$$\frac{I\left(l_{ij}^{k-}\right)}{I\left(l_{ji}^{k+}\right)} w_j \leqslant w_i \leqslant \frac{I\left(l_{ij}^{k+}\right)}{I\left(l_{ji}^{k-}\right)} w_j,\ \text{for all } \delta_{ij}^k = 1,\ i, j \in N,\ i \neq j,\ k \in K_3, \tag{4.15}$$

or

$$\frac{1}{\sum_{\substack{j=1\\ j\neq i}}^{n} \delta_{ij}^k} \sum_{\substack{j=1\\ j\neq i}}^{n} \delta_{ij}^k \frac{I\left(l_{ij}^{k-}\right)}{I\left(l_{ji}^{k+}\right)} w_j \leqslant w_i \leqslant \frac{1}{\sum_{\substack{j=1\\ j\neq i}}^{n} \delta_{ij}^k} \sum_{\substack{j=1\\ j\neq i}}^{n} \delta_{ij}^k \frac{I\left(l_{ij}^{k+}\right)}{I\left(l_{ji}^{k-}\right)} w_j,\ i \in N. \tag{4.16}$$

4.1.4. Intuitionistic fuzzy preference relations

By Definition 7, if the intuitionistic fuzzy preference relation provided by the kth decision maker $(k \in K_4)$ is consistent with the group's opinion, we have

$$u_{ij}^k \leqslant \frac{w_i}{w_i + w_j} \leqslant 1 - \upsilon_{ij}^k, \text{ for all } \delta_{ij}^k = 1, \ i, \ j \in N, \ i \neq j, \ k \in K_4, \tag{4.17}$$

i.e.

$$\frac{u_{ij}^k}{1 - u_{ij}^k} w_j \leqslant w_i \leqslant \frac{1 - \upsilon_{ij}^k}{\upsilon_{ij}^k} w_j, \text{ for all } \delta_{ij}^k = 1, \ i, \ j \in N, \ i \neq j, \ k \in K_4. \tag{4.18}$$

Equivalently,

$$\delta_{ij}^k \frac{u_{ij}^k}{1 - u_{ij}^k} w_j \leqslant \delta_{ij}^k w_i \leqslant \delta_{ij}^k \frac{1 - \upsilon_{ij}^k}{\upsilon_{ij}^k} w_j, \ i, \ j \in N, \ i \neq j, \ k \in K_4. \tag{4.19}$$

Summing all the entries of Eq. (4.19) for all $j = 1, \ 2, \ \ldots, \ n, \ j \neq i$, we have

$$\sum_{\substack{j=1 \\ j \neq i}}^n \delta_{ij}^k \frac{u_{ij}^k}{1 - u_{ij}^k} w_j \leqslant \sum_{\substack{j=1 \\ j \neq i}}^n \delta_{ij}^k w_i \leqslant \sum_{\substack{j=1 \\ j \neq i}}^n \delta_{ij}^k \frac{1 - \upsilon_{ij}^k}{\upsilon_{ij}^k} w_j, \ i \in N, \ k \in K_4. \tag{4.20}$$

As $\sum_{\substack{j=1 \\ j \neq i}}^n \delta_{ij}^k \neq 0$, it follows that

$$\frac{1}{\sum_{\substack{j=1 \\ j \neq i}}^n \delta_{ij}^k} \sum_{\substack{j=1 \\ j \neq i}}^n \delta_{ij}^k \frac{u_{ij}^k}{1 - u_{ij}^k} w_j \leqslant w_i \leqslant \frac{1}{\sum_{\substack{j=1 \\ j \neq i}}^n \delta_{ij}^k} \sum_{\substack{j=1 \\ j \neq i}}^n \delta_{ij}^k \frac{1 - \upsilon_{ij}^k}{\upsilon_{ij}^k} w_j, \ i \in N. \tag{4.21}$$

4.1.5. The group consensus index

Let

$$a_{ij}^{k-} = \frac{b_{ij}^{k-}}{b_{ji}^{k+}}, \ a_{ij}^{k+} = \frac{b_{ij}^{k+}}{b_{ji}^{k-}}, \text{ for all } \delta_{ij}^k = 1, \ i, \ j \in N, \ i \neq j, \ k \in K_2, \tag{4.22}$$

$$a_{ij}^{k-} = \frac{I\left(l_{ij}^{k-}\right)}{I\left(l_{ji}^{k+}\right)}, \ a_{ij}^{k+} = \frac{I\left(l_{ij}^{k+}\right)}{I\left(l_{ji}^{k-}\right)}, \text{ for all } \delta_{ij}^k = 1, \ i, \ j \in N, \ i \neq j, \ k \in K_3, \tag{4.23}$$

$$a_{ij}^{k-} = \frac{u_{ij}^k}{1 - u_{ij}^k}, \ a_{ij}^{k+} = \frac{1 - \upsilon_{ij}^k}{\upsilon_{ij}^k}, \text{ for all } \delta_{ij}^k = 1, \ i, \ j \in N, \ i \neq j, \ k \in K_4. \tag{4.24}$$

By Eqs. (4.5), (4.11), (4.16) and (4.21), the priority weights obtained from the kth decision maker's preference relation can be denoted as

$$\tilde{w}_i^k = [w_i^{k-}, w_i^{k+}] = \left[\frac{1}{\sum_{\substack{j=1 \\ j \neq i}}^n \delta_{ij}^k} \sum_{\substack{j=1 \\ j \neq i}}^n \delta_{ij}^k a_{ij}^{k-} w_j, \frac{1}{\sum_{\substack{j=1 \\ j \neq i}}^n \delta_{ij}^k} \sum_{\substack{j=1 \\ j \neq i}}^n \delta_{ij}^k a_{ij}^{k+} w_j\right], \ i \in N, \ k \in K. \tag{4.25}$$

Therefore, the deviation between the kth decision maker's opinion and the lth decision maker's opinion can be calculated as

$$d(k, l) = \sum_{i=1}^n \frac{1}{2} \left(|w_i^{k-} - w_i^{l-}| + |w_i^{k+} - w_i^{l+}|\right), \ k, \ l \in K, \ k \neq l. \tag{4.26}$$

Then, the deviation between the kth decision maker's opinion and other decision makers' opinions can be calculated as

$$d_k = \sum_{\substack{l=1 \\ l \neq k}}^q d(k, l) = \sum_{\substack{l=1 \\ l \neq k}}^q \sum_{i=1}^n \frac{1}{2} \left(|w_i^{k-} - w_i^{l-}| + |w_i^{k+} - w_i^{l+}|\right), \ k \in K. \tag{4.27}$$

If we consider the importance of each decision maker, then the overall deviation can be obtained as

$$d = \sum_{k=1}^q \lambda_k d_k = \sum_{k=1}^q \sum_{\substack{l=1 \\ l \neq k}}^q \sum_{i=1}^n \frac{1}{2} \lambda_k \left(|w_i^{k-} - w_i^{l-}| + |w_i^{k+} - w_i^{l+}|\right) = \sum_{k=1}^{q-1} \sum_{l=k+1}^q \sum_{i=1}^n \frac{1}{2} (\lambda_k + \lambda_l) \left(|w_i^{k-} - w_i^{l-}| + |w_i^{k+} - w_i^{l+}|\right). \tag{4.28}$$

By Eq. (4.25), Eq. (4.28) can be rewritten as

$$d = \frac{1}{2} \sum_{k=1}^{q-1} \sum_{l=k+1}^q \sum_{i=1}^n (\lambda_k + \lambda_l) \left(\left| \frac{1}{\sum_{\substack{j=1 \\ j \neq i}}^n \delta_{ij}^k} \sum_{\substack{j=1 \\ j \neq i}}^n \delta_{ij}^k a_{ij}^{k-} w_j - \frac{1}{\sum_{\substack{j=1 \\ j \neq i}}^n \delta_{ij}^l} \sum_{\substack{j=1 \\ j \neq i}}^n \delta_{ij}^l a_{ij}^{l-} w_j \right| + \left| \frac{1}{\sum_{\substack{j=1 \\ j \neq i}}^n \delta_{ij}^k} \sum_{\substack{j=1 \\ j \neq i}}^n \delta_{ij}^k a_{ij}^{k+} w_j - \frac{1}{\sum_{\substack{j=1 \\ j \neq i}}^n \delta_{ij}^l} \sum_{\substack{j=1 \\ j \neq i}}^n \delta_{ij}^l a_{ij}^{l+} w_j \right| \right). \tag{4.29}$$

Here we call Eq. (4.29) the group consensus index. To obtain good consensus for the group decision making problem, the value of d should be minimized.

4.2. Calculation of the overall consistency index

For decision making problems with preference relations, consistency is also quite important, since inconsistent judgments will result in unreasonable decision result. For the group decision making problem to be solved, if the preference relations provided by the decision makers are consistent, it follows that Eqs. (4.2), (4.7), (4.15) and (4.18) hold. Based on the definitions of the four formats of uncertain preference relations, Eqs. (4.2), (4.7), (4.15) and (4.18) can be further written as

$$a_{ij}^{k-} w_j \leqslant w_i \leqslant a_{ij}^{k+} w_j, \text{ for all } \delta_{ij}^k = 1, \ i, \ j \in N, \ i < j, \ k \in K_1, \tag{4.30}$$

$$\frac{b_{ij}^{k-}}{b_{ji}^{k+}} w_j \leqslant w_i \leqslant \frac{b_{ij}^{k+}}{b_{ji}^{k-}} w_j, \text{ for all } \delta_{ij}^k = 1, \ i, \ j \in N, \ i < j, \ k \in K_2. \tag{4.31}$$

$$\frac{I(l_{ij}^{k-})}{I(l_{ji}^{k+})} w_j \leqslant w_i \leqslant \frac{I(l_{ij}^{k+})}{I(l_{ji}^{k-})} w_j, \text{ for all } \delta_{ij}^k = 1, \ i, \ j \in N, \ i < j, \ k \in K_3, \tag{4.32}$$

$$\frac{u_{ij}^k}{1 - u_{ij}^k} w_j \leqslant w_i \leqslant \frac{1 - \upsilon_{ij}^k}{\upsilon_{ij}^k} w_j, \text{ for all } \delta_{ij}^k = 1, \ i, \ j \in N, \ i < j, \ k \in K_4. \tag{4.33}$$

By Eqs. (4.22), (4.23) and (4.24), Eqs. (4.30), (4.31), (4.32) and (4.33) can be rewritten as

$$a_{ij}^{k-} w_j \leqslant w_i \leqslant a_{ij}^{k+} w_j, \text{ for all } \delta_{ij}^k = 1, \ i, \ j \in N, \ i < j, \ k \in K. \tag{4.34}$$

However, decision makers' preferences are not completely consistent in most cases. As a result, Eq. (4.34) will not hold simultaneously. Moreover, the preference over some alternatives from different decision makers may be conflicting. To deal with such situations, we can relax Eq. (4.34) by introducing the deviation variables as follows:

$$a_{ij}^{k-} w_j - \eta_{ij}^{k-} \leqslant w_i \leqslant a_{ij}^{k+} w_j + \eta_{ij}^{k+}, \text{ for all } \delta_{ij}^k = 1, \ i, j \in N, \ i < j, \ k \in K, \tag{4.35}$$

where both η_{ij}^{k-} and η_{ij}^{k+} are non-negative real numbers.

Taking the importance of the decision makers into account, the overall deviation for all the individual preference relations can be calculated as

$$s = \sum_{k=1}^{q} \lambda_k \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \delta_{ij}^{k} \left(\eta_{ij}^{k-} + \eta_{ij}^{k+} \right). \tag{4.36}$$

It is obvious that the smaller the value of s, the higher the consistency of the given preference relations. To obtain higher consistency, the value of s should also be minimized.

4.3. Establishment of optimization models for deriving the priority weight vector

As mentioned above, to obtain a more reasonable result, both the group consensus index and the overall consistency index should be minimized. Based on Eqs. (4.29) and (4.36), the following optimization model can be established:

$$\begin{aligned}
\min\ Z_1 = &\frac{1}{2} \sum_{k=1}^{q-1} \sum_{l=k+1}^{q} \sum_{i=1}^{n} (\lambda_k + \lambda_l) \\
&\left(\left| \frac{1}{\sum_{j=1, j\neq i}^{n} \delta_{ij}^{k}} \sum_{j=1, j\neq i}^{n} \delta_{ij}^{k} a_{ij}^{k-} w_j - \frac{1}{\sum_{j=1, j\neq i}^{n} \delta_{ij}^{l}} \sum_{j=1, j\neq i}^{n} \delta_{ij}^{l} a_{ij}^{l-} w_j \right| \right. \\
&\left. + \left| \frac{1}{\sum_{j=1, j\neq i}^{n} \delta_{ij}^{k}} \sum_{j=1, j\neq i}^{n} \delta_{ij}^{k} a_{ij}^{k+} w_j - \frac{1}{\sum_{j=1, j\neq i}^{n} \delta_{ij}^{l}} \sum_{j=1, j\neq i}^{n} \delta_{ij}^{l} a_{ij}^{l+} w_j \right| \right) \\
\min\ Z_2 = &\sum_{k=1}^{q} \lambda_k \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \delta_{ij}^{k} \left(\eta_{ij}^{k-} + \eta_{ij}^{k+} \right) \\
\text{s.t.}\ & a_{ij}^{k-} w_j - \eta_{ij}^{k-} \leqslant w_i \leqslant a_{ij}^{k+} w_j + \eta_{ij}^{k+},\ \text{for all } \delta_{ij}^{k} = 1,\ i, j \in N,\ i < j,\ k \in K \\
& \sum_{i=1}^{n} w_i = 1,\ w_i > 0,\ i \in N \\
& \eta_{ij}^{k-},\ \eta_{ij}^{k+} \geqslant 0,\ \text{for all } \delta_{ij}^{k} = 1,\ i, j \in N,\ i < j,\ k \in K.
\end{aligned} \tag{M-1}$$

Let

$$\begin{aligned}
\mu_i^{kl+} = \frac{1}{2} \Bigg[&\left| \frac{1}{\sum_{j=1, j\neq i}^{n} \delta_{ij}^{k}} \sum_{j=1, j\neq i}^{n} \delta_{ij}^{k} a_{ij}^{k-} w_j - \frac{1}{\sum_{j=1, j\neq i}^{n} \delta_{ij}^{l}} \sum_{j=1, j\neq i}^{n} \delta_{ij}^{l} a_{ij}^{l-} w_j \right| \\
&+ \left(\frac{1}{\sum_{j=1, j\neq i}^{n} \delta_{ij}^{k}} \sum_{j=1, j\neq i}^{n} \delta_{ij}^{k} a_{ij}^{k-} w_j - \frac{1}{\sum_{j=1, j\neq i}^{n} \delta_{ij}^{l}} \sum_{j=1, j\neq i}^{n} \delta_{ij}^{l} a_{ij}^{l-} w_j \right) \Bigg],
\end{aligned} \tag{4.37}$$

$$\begin{aligned}
\mu_i^{kl-} = \frac{1}{2} \Bigg[&\left| \frac{1}{\sum_{j=1, j\neq i}^{n} \delta_{ij}^{k}} \sum_{j=1, j\neq i}^{n} \delta_{ij}^{k} a_{ij}^{k-} w_j - \frac{1}{\sum_{j=1, j\neq i}^{n} \delta_{ij}^{l}} \sum_{j=1, j\neq i}^{n} \delta_{ij}^{l} a_{ij}^{l-} w_j \right| \\
&- \left(\frac{1}{\sum_{j=1, j\neq i}^{n} \delta_{ij}^{k}} \sum_{j=1, j\neq i}^{n} \delta_{ij}^{k} a_{ij}^{k-} w_j - \frac{1}{\sum_{j=1, j\neq i}^{n} \delta_{ij}^{l}} \sum_{j=1, j\neq i}^{n} \delta_{ij}^{l} a_{ij}^{l-} w_j \right) \Bigg],
\end{aligned} \tag{4.38}$$

$$\begin{aligned}
v_i^{kl+} = \frac{1}{2} \Bigg[&\left| \frac{1}{\sum_{j=1, j\neq i}^{n} \delta_{ij}^{k}} \sum_{j=1, j\neq i}^{n} \delta_{ij}^{k} a_{ij}^{k+} w_j - \frac{1}{\sum_{j=1, j\neq i}^{n} \delta_{ij}^{l}} \sum_{j=1, j\neq i}^{n} \delta_{ij}^{l} a_{ij}^{l+} w_j \right| \\
&+ \left(\frac{1}{\sum_{j=1, j\neq i}^{n} \delta_{ij}^{k}} \sum_{j=1, j\neq i}^{n} \delta_{ij}^{k} a_{ij}^{k+} w_j - \frac{1}{\sum_{j=1, j\neq i}^{n} \delta_{ij}^{l}} \sum_{j=1, j\neq i}^{n} \delta_{ij}^{l} a_{ij}^{l+} w_j \right) \Bigg],
\end{aligned} \tag{4.39}$$

$$\begin{aligned}
v_i^{kl-} = \frac{1}{2} \Bigg[&\left| \frac{1}{\sum_{j=1, j\neq i}^{n} \delta_{ij}^{k}} \sum_{j=1, j\neq i}^{n} \delta_{ij}^{k} a_{ij}^{k+} w_j - \frac{1}{\sum_{j=1, j\neq i}^{n} \delta_{ij}^{l}} \sum_{j=1, j\neq i}^{n} \delta_{ij}^{l} a_{ij}^{l+} w_j \right| \\
&- \left(\frac{1}{\sum_{j=1, j\neq i}^{n} \delta_{ij}^{k}} \sum_{j=1, j\neq i}^{n} \delta_{ij}^{k} a_{ij}^{k+} w_j - \frac{1}{\sum_{j=1, j\neq i}^{n} \delta_{ij}^{l}} \sum_{j=1, j\neq i}^{n} \delta_{ij}^{l} a_{ij}^{l+} w_j \right) \Bigg], \\
&\text{for all } i \in N,\ k,\ l \in K,\ k < l.
\end{aligned} \tag{4.40}$$

Then the model (M-1) can be transformed into

$$\begin{aligned}
\min\ \ & Z_1 = \frac{1}{2} \sum_{k=1}^{q-1} \sum_{l=k+1}^{q} \sum_{i=1}^{n} (\lambda_k + \lambda_l) \left(\mu_i^{kl-} + \mu_i^{kl+} + v_i^{kl-} + v_i^{kl+} \right) \\
\min\ \ & Z_2 = \sum_{k=1}^{q} \lambda_k \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \delta_{ij}^{k} \left(\eta_{ij}^{k-} + \eta_{ij}^{k+} \right) \\
\text{s.t.}\ \ & a_{ij}^{k-} w_j - \eta_{ij}^{k-} \leqslant w_i \leqslant a_{ij}^{k+} w_j + \eta_{ij}^{k+}, \\
& \quad \text{for all } \delta_{ij}^{k} = 1,\ i,\ j \in N,\ i < j,\ k \in K \\
& \frac{1}{\sum_{j=1, j\neq i}^{n} \delta_{ij}^{k}} \sum_{j=1, j\neq i}^{n} \delta_{ij}^{k} a_{ij}^{k-} w_j - \frac{1}{\sum_{j=1, j\neq i}^{n} \delta_{ij}^{l}} \sum_{j=1, j\neq i}^{n} \delta_{ij}^{l} a_{ij}^{l-} w_j \\
& \quad + \mu_i^{kl-} - \mu_i^{kl+} = 0,\ i \in N,\ k,\ l \in K,\ k < l \\
& \frac{1}{\sum_{j=1, j\neq i}^{n} \delta_{ij}^{k}} \sum_{j=1, j\neq i}^{n} \delta_{ij}^{k} a_{ij}^{k+} w_j - \frac{1}{\sum_{j=1, j\neq i}^{n} \delta_{ij}^{l}} \sum_{j=1, j\neq i}^{n} \delta_{ij}^{l} a_{ij}^{l+} w_j \\
& \quad + v_i^{kl-} - v_i^{kl+} = 0\ i \in N,\ k,\ l \in K,\ k < l \\
& \sum_{i=1}^{n} w_i = 1,\ w_i > 0,\ i \in N \\
& \eta_{ij}^{k-},\ \eta_{ij}^{k+} \geqslant 0,\ \text{for all } \delta_{ij}^{k} = 1,\ i,\ j \in N,\ i < j,\ k \in K \\
& \mu_i^{kl-},\ \mu_i^{kl+},\ v_i^{kl-},\ v_i^{kl+} \geqslant 0,\ i \in N,\ k,\ l \in K,\ k < l.
\end{aligned} \tag{M-2}$$

The model (M-1) is a linear bi-objective programming model. In what follows, we utilize Zimmermann's max–min approach (Zimmermann, 1978) to solve the model. First, we solve the following optimization models:

$$\begin{aligned}
\min / \max\ \ & Z_1 = \frac{1}{2} \sum_{k=1}^{q-1} \sum_{l=k+1}^{q} \sum_{i=1}^{n} (\lambda_k + \lambda_l) \left(\mu_i^{kl-} + \mu_i^{kl+} + v_i^{kl-} + v_i^{kl+} \right) \\
\text{s.t.}\ \ & a_{ij}^{k-} w_j - \eta_{ij}^{k-} \leqslant w_i \leqslant a_{ij}^{k+} w_j + \eta_{ij}^{k+}, \\
& \text{for all } \delta_{ij}^{k} = 1,\ i,\ j \in N,\ i < j,\ k \in K \\
& \frac{1}{\sum_{j=1, j\neq i}^{n} \delta_{ij}^{k}} \sum_{j=1, j\neq i}^{n} \delta_{ij}^{k} a_{ij}^{k-} w_j - \frac{1}{\sum_{j=1, j\neq i}^{n} \delta_{ij}^{l}} \sum_{j=1, j\neq i}^{n} \delta_{ij}^{l} a_{ij}^{l-} w_j \\
& + \mu_i^{kl-} - \mu_i^{kl+} = 0,\ i \in N,\ k,\ l \in K,\ k < l \\
& \frac{1}{\sum_{j=1, j\neq i}^{n} \delta_{ij}^{k}} \sum_{j=1, j\neq i}^{n} \delta_{ij}^{k} a_{ij}^{k+} w_j - \frac{1}{\sum_{j=1, j\neq i}^{n} \delta_{ij}^{l}} \sum_{j=1, j\neq i}^{n} \delta_{ij}^{l} a_{ij}^{l+} w_j \\
& + v_i^{kl-} - v_i^{kl+} = 0,\ i \in N,\ k,\ l \in K,\ k < l \\
& \sum_{i=1}^{n} w_i = 1,\ w_i > 0,\ i \in N, \quad \eta_{ij}^{k-},\ \eta_{ij}^{k+} \geqslant 0, \\
& \text{for all } \delta_{ij}^{k} = 1,\ i,\ j \in N,\ i < j,\ k \in k \\
& \mu_i^{kl-},\ \mu_i^{kl+},\ v_i^{kl-},\ v_i^{kl+} \geqslant 0,\ i \in N,\ k,\ l \in K,\ k < l
\end{aligned} \tag{M-3}$$

and

$$
\begin{aligned}
\min/\max\ & Z_2=\sum_{k=1}^{q}\lambda_k\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\delta_{ij}^{k}\left(\eta_{ij}^{k-}+\eta_{ij}^{k+}\right)\\
\text{s.t.}\ & a_{ij}^{k-}w_j-\eta_{ij}^{k-}\leqslant w_i\leqslant a_{ij}^{k+}w_j+\eta_{ij}^{k+},\ \text{for all}\ \delta_{ij}^{k}=1,\ i,j\in N,\ i<j,\ k\in K\\
& \frac{1}{\sum_{j=1,j\neq i}^{n}\delta_{ij}^{k}}\sum_{j=1,j\neq i}^{n}\delta_{ij}^{k}a_{ij}^{k-}w_j-\frac{1}{\sum_{j=1,j\neq i}^{n}\delta_{ij}^{l}}\sum_{j=1,j\neq i}^{n}\delta_{ij}^{l}a_{ij}^{l-}w_j\\
& +\mu_i^{kl-}-\mu_i^{kl+}=0,\ i\in N,\ k,\ l\in K,\ k<l\\
& \frac{1}{\sum_{j=1,j\neq i}^{n}\delta_{ij}^{k}}\sum_{j=1,j\neq i}^{n}\delta_{ij}^{k}a_{ij}^{k+}w_j-\frac{1}{\sum_{j=1,j\neq i}^{n}\delta_{ij}^{l}}\sum_{j=1,j\neq i}^{n}\delta_{ij}^{l}a_{ij}^{l+}w_j\\
& +\nu_i^{kl-}-\nu_i^{kl+}=0,\ i\in N,\ k,\ l\in K,\ k<l\\
& \sum_{i=1}^{n}w_i=1,\ w_i>0,\ i\in N\\
& \eta_{ij}^{k-},\ \eta_{ij}^{k+}\geqslant 0,\ \text{for all}\ \delta_{ij}^{k}=1,\ i,j\in N,\ i<j,\ k\in K\\
& \mu_i^{kl-},\ \mu_i^{kl+},\ \nu_i^{kl-},\ \nu_i^{kl+}\geqslant 0,\ i\in N,\ k,\ l\in K,\ k<l.
\end{aligned}
\tag{M-4}
$$

The models (M-3) and (M-4) are simple linear programming models, which can be easily solved with the use of some optimization software packages. Let Z_1^{min} and Z_1^{max} be the minimum objective function value and the maximum objective function value derived from the model (M-3), Z_2^{min} and Z_2^{max} be the minimum objective function value and the maximum objective function value derived from the model (M-4), then we can construct the following membership functions as

$$
\mu_1(Z_1)=\begin{cases}1 & \text{if } Z_1<Z_1^{min}\\ \frac{Z_1^{max}-Z_1}{Z_1^{max}-Z_1^{min}} & \text{if } Z_1^{min}\leqslant Z_1\leqslant Z_1^{max}\\ 0 & \text{if } Z_1>Z_1^{max},\end{cases}
\tag{4.41}
$$

and

$$
\mu_2(Z_2)=\begin{cases}1 & \text{if } Z_2<Z_2^{min}\\ \frac{Z_2^{max}-Z_2}{Z_2^{max}-Z_2^{min}} & \text{if } Z_2^{min}\leqslant Z_2\leqslant Z_2^{max}\\ 0 & \text{if } Z_2>Z_2^{max}.\end{cases}
\tag{4.42}
$$

By Zimmermann's max–min approach (Zimmermann, 1978), the model (M-1) can be transformed into the following model:

$$
\begin{aligned}
\max\ & \theta\\
\text{s.t.}\ & \mu_1(Z_1)\geqslant\theta\\
& \mu_2(Z_2)\geqslant\theta\\
& a_{ij}^{k-}w_j-\eta_{ij}^{k-}\leqslant w_i\leqslant a_{ij}^{k+}w_j+\eta_{ij}^{k+},\ \text{for all}\ \delta_{ij}^{k}=1,\ i,j\in N,\ i<j,\ k\in K\\
& \frac{1}{\sum_{j=1,j\neq i}^{n}\delta_{ij}^{k}}\sum_{j=1,j\neq i}^{n}\delta_{ij}^{k}a_{ij}^{k-}w_j-\frac{1}{\sum_{j=1,j\neq i}^{n}\delta_{ij}^{l}}\sum_{j=1,j\neq i}^{n}\delta_{ij}^{l}a_{ij}^{l-}w_j\\
& +\mu_i^{kl-}-\mu_i^{kl+}=0,\ i\in N,\ k,\ l\in K,\ k<l\\
& \frac{1}{\sum_{j=1,j\neq i}^{n}\delta_{ij}^{k}}\sum_{j=1,j\neq i}^{n}\delta_{ij}^{k}a_{ij}^{k+}w_j-\frac{1}{\sum_{j=1,j\neq i}^{n}\delta_{ij}^{l}}\sum_{j=1,j\neq i}^{n}\delta_{ij}^{l}a_{ij}^{l+}w_j\\
& +\nu_i^{kl-}-\nu_i^{kl+}=0,\ ,\ i\in N,\ k,\ l\in K,\ k<l\\
& \sum_{i=1}^{n}w_i=1,\ w_i>0,\ i\in N\\
& \eta_{ij}^{k-},\ \eta_{ij}^{k+}\geqslant 0,\ \text{for all}\ \delta_{ij}^{k}=1,\ i,j\in N,\ i<j,\ k\in K\\
& \mu_i^{kl-},\ \mu_i^{kl+},\ \nu_i^{kl-},\ \nu_i^{kl+}\geqslant 0,\ i\in N,\ k,\ l\in K,\ k<l.
\end{aligned}
\tag{M-5}
$$

By solving the model (M-5), the priority weights can be obtained as $w=(w_1,w_2,\ldots,w_n)^{\mathrm{T}}$ and the value of θ reflects the minimum satisfactory level of the two objective functions. To summarize, we present the group decision making approach as follows.

Step 1: Obtain the indication matrices Δ^k, $k\in K$ by Eqs. (3.1a), (3.1b), (3.1c) and (3.1d).

Step 2: Calculate the value of a_{ij}^{k-} and a_{ij}^{k+} for all $\delta_{ij}^{k}=1$, $i,\ j\in N$, $i\neq j$, $k\in K_2\cup K_3\cup K_4$ by Eqs. (4.22), (4.23) and (4.24).

Step 3: Solve the model (M-3) to obtain Z_1^{min} and Z_1^{max}.

Step 4: Solve the model (M-4) to obtain Z_2^{min} and Z_2^{max}.

Step 5: Construct the membership functions for Z_1 and Z_2 by Eqs. (4.41) and (4.42), respectively, and then solve the model (M-5) to derive the priority weight vector $w=(w_1,w_2,\ldots,w_n)^{\mathrm{T}}$.

Step 6: Based on w, rank the alternatives and select the best alternative.

5. Illustrative examples

In this section, we give some examples to illustrate the feasibility and effectiveness of the proposed method. First, we consider the selection of investment alternatives in a multinational corporation.

A multinational corporation intends to invest a sum of money and there are five alternatives $(x_1,x_2,\ldots,x_5)$ to be selected. Four director board members (d_1,d_2,d_3,d_4) are invited to evaluate the five investment alternatives and the weight vector of the four director board members is $\lambda=(1/4,1/4,1/4,1/4)^{\mathrm{T}}$. Due to the difference of culture and education backgrounds, they provide their preference information on the five alternatives through pairwise comparisons using different formats of uncertain preference relations. Specifically, the preference relations provided by the four director board members are uncertain multiplicative preference relation, uncertain fuzzy preference relation, uncertain linguistic preference relation and intuitionistic fuzzy preference relation, respectively. The linguistic terms sets used by d_3 is $S=\{s_0:\text{very poor},\ s_1:\text{poor},s_2:\text{slightly poor},s_3:\text{fair},s_4:\text{slightly good},\ s_5:\text{good},\ s_6:\text{very good}\}$. Due to the lack of knowledge, the preference relations provided by them are incomplete ones as follows.

$$
\tilde{A}_1=\begin{pmatrix}[1,1] & [1/3,3] & \varphi & [5,7] & [3,5]\\ [1/3,3] & [1,1] & [1,3] & \varphi & [1,3]\\ \varphi & [1/3,1] & [1,1] & [5,7] & [1,3]\\ [1/7,1/5] & \varphi & [1/7,1/5] & [1,1] & \varphi\\ [1/5,1/3] & [1/3,1] & [1/3,1] & \varphi & [1,1]\end{pmatrix},
$$

$$
\tilde{B}_2=\begin{pmatrix}[0.5,0.5] & \varphi & [0.4,0.6] & [0.5,0.7] & [0.3,0.5]\\ \varphi & [0.5,0.5] & [0.3,0.6] & \varphi & [0.5,0.8]\\ [0.4,0.6] & [0.4,0.7] & [0.5,0.5] & [0.6,0.8] & \varphi\\ [0.3,0.5] & \varphi & [0.2,0.4] & [0.5,0.5] & \varphi\\ [0.5,0.7] & [0.2,0.5] & \varphi & \varphi & [0.5,0.5]\end{pmatrix},
$$

$$
\tilde{L}_3=\begin{pmatrix}[s_3,s_3] & \varphi & [s_3,s_5] & [s_3,s_5] & [s_4,s_4]\\ \varphi & [s_3,s_3] & \varphi & [s_3,s_4] & [s_3,s_4]\\ [s_1,s_3] & \varphi & [s_3,s_3] & [s_3,s_4] & [s_4,s_5]\\ [s_1,s_3] & [s_2,s_3] & [s_2,s_3] & [s_3,s_3] & \varphi\\ [s_2,s_2] & [s_2,s_3] & [s_1,s_2] & \varphi & [s_3,s_3]\end{pmatrix},
$$

$$\tilde{R}_4 = \begin{pmatrix} (0.5, 0.5) & (0.6, 0.1) & (0.8, 0.2) & (0.6, 0.3) & (0.7, 0.2) \\ (0.1, 0.6) & (0.5, 0.5) & (0.5, 0.1) & \varphi & (0.6, 0.1) \\ (0.2, 0.8) & (0.1, 0.5) & (0.5, 0.5) & (0.4, 0.6) & \varphi \\ (0.3, 0.6) & \varphi & (0.6, 0.4) & (0.5, 0.5) & (0.7, 0.3) \\ (0.2, 0.7) & (0.1, 0.6) & \varphi & (0.3, 0.7) & (0.5, 0.5) \end{pmatrix},$$

In what follows, we will utilize the proposed method to select the best investment alternative.

By Eqs. (3.1a)–(3.1d), we have

$$\Delta^1 = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix}, \quad \Delta^2 = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix},$$

$$\Delta^3 = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix}, \quad \Delta^4 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

Then, by Eqs. (4.22), (4.23) and (4.24), we can calculate $a_{13}^{2-} = a_{31}^{2-} = a_{32}^{2-} = 2/3$, $a_{14}^{2-} = a_{25}^{2-} = a_{51}^{2-} = 1$, $a_{15}^{2-} = a_{23}^{2-} = a_{41}^{2-} = 3/7$, $a_{34}^{2-} = 1.5$, $a_{43}^{2-} = a_{52}^{2-} = 0.25$; $a_{13}^{2+} = a_{23}^{2+} = a_{31}^{2+} = 1.5$, $a_{14}^{2+} = a_{32}^{2+} = a_{51}^{2+} = 7/3$, $a_{15}^{2+} = a_{41}^{2+} = a_{52}^{2+} = 1$, $a_{25}^{2+} = a_{34}^{2+} = 4$, $a_{43}^{2+} = 2/3$; $a_{13}^{3-} = a_{14}^{3-} = a_{24}^{3-} = a_{25}^{3-} = a_{34}^{3-} = 1$, $a_{15}^{3-} = a_{35}^{3-} = 2$, $a_{31}^{3-} = a_{41}^{3-} = a_{53}^{3-} = 0.2$, $a_{42}^{3-} = a_{43}^{3-} = a_{51}^{3-} = a_{52}^{3-} = 0.5$; $a_{13}^{3+} = a_{14}^{3+} = a_{35}^{3+} = 5$, $a_{15}^{3+} = a_{24}^{3+} = a_{25}^{3+} = a_{34}^{3+} = 2$, $a_{31}^{3+} = a_{41}^{3+} = a_{42}^{3+} = a_{43}^{3+} = a_{52}^{3+} = 1$, $a_{51}^{3+} = a_{53}^{3+} = 0.5$; $a_{12}^{4-} = a_{14}^{4-} = a_{25}^{4-} = a_{43}^{4-} = 1.5$, $a_{13}^{4-} = 4$, $a_{15}^{4-} = a_{45}^{4-} = 7/3$, $a_{21}^{4-} = a_{32}^{4-} = a_{52}^{4-} = 1/9$, $a_{23}^{4-} = 1$, $a_{31}^{4-} = a_{51}^{4-} = 0.25$, $a_{34}^{4-} = 2/3$, $a_{41}^{4-} = a_{54}^{4-} = 3/7$; $a_{12}^{4+} = a_{23}^{4+} = a_{25}^{4+} = 9$, $a_{13}^{4+} = a_{15}^{4+} = 4$, $a_{14}^{4+} = a_{45}^{4+} = 7/3$, $a_{21}^{4+} = a_{34}^{4+} = a_{41}^{4+} = a_{52}^{4+} = 2/3$, $a_{31}^{4+} = 0.25$, $a_{32}^{4+} = 1$, $a_{43}^{4+} = 1.5$, $a_{51}^{4+} = a_{54}^{4+} = 3/7$.

By solving the models (M-3), (M-4) and (M-5) (the models are omitted here due to the limitations of space), the priority weights can be derived as $w_1 = 0.4505$, $w_2 = 0.1978$, $w_3 = 0.1319$, $w_4 = 0.0989$, $w_5 = 0.1209$ and $\theta = 0.9963$. Therefore, the ranking of the five alternatives is $x_1 \succ x_2 \succ x_3 \succ x_5 \succ x_4$ and x_1 is the best investment alternative. The value of θ shows that the satisfactory levels of the two objective functions are larger than 0.9963, which means that both the group consensus and individual consistency are sufficiently considered.

In the rest of this section, we make a comparison between the proposed approach and existing approaches to further verify the effectiveness of the proposed method. As there is no work focusing on group decision making with different formats of incomplete uncertain preference relations to the best of our knowledge, we consider a special case of the group decision making problem, i.e. group decision making with incomplete uncertain fuzzy preference relations. Xia and Xu (2011) proposed a procedure to estimate the missing elements for incomplete uncertain fuzzy preference relations and developed an approach to group decision making with incomplete uncertain fuzzy preference relations. In Xia and Xu (2011), four decision makers (the weight vector is $\lambda = (1/4, 1/4, 1/4, 1/4)^{\mathrm{T}}$) gave their preference information over four alternatives $X = \{x_1, x_2, x_3, x_4\}$ using incomplete uncertain fuzzy preference relations as follows.

$$\widetilde{B}_1 = \begin{pmatrix} [0.5, 0.5] & [0.4, 0.6] & [0.6, 0.8] & [0.3, 0.5] \\ [0.4, 0.6] & [0.5, 0.5] & \varphi & [0.4, 0.7] \\ [0.2, 0.4] & \varphi & [0.5, 0.5] & [0.3, 0.4] \\ [0.5, 0.7] & [0.3, 0.6] & [0.6, 0.7] & [0.5, 0.5] \end{pmatrix},$$

$$\widetilde{B}_2 = \begin{pmatrix} [0.5, 0.5] & [0.5, 0.7] & [0.3, 0.6] & \varphi \\ [0.3, 0.5] & [0.5, 0.5] & [0.4, 0.5] & [0.1, 0.2] \\ [0.4, 0.7] & [0.5, 0.6] & [0.5, 0.5] & [0.3, 0.4] \\ \varphi & [0.8, 0.9] & [0.6, 0.7] & [0.5, 0.5] \end{pmatrix},$$

$$\widetilde{B}_3 = \begin{pmatrix} [0.5, 0.5] & [0.6, 0.9] & \varphi & [0.4, 0.6] \\ [0.1, 0.4] & [0.5, 0.5] & [0.6, 0.8] & [0.3, 0.5] \\ \varphi & [0.2, 0.4] & [0.5, 0.5] & [0.5, 0.6] \\ [0.4, 0.6] & [0.5, 0.7] & [0.4, 0.5] & [0.5, 0.5] \end{pmatrix},$$

$$\widetilde{B}_4 = \begin{pmatrix} [0.5, 0.5] & [0.2, 0.4] & [0.3, 0.5] & [0.4, 0.6] \\ [0.6, 0.8] & [0.5, 0.5] & [0.5, 0.6] & [0.6, 0.7] \\ [0.5, 0.7] & [0.4, 0.5] & [0.5, 0.5] & \varphi \\ [0.4, 0.6] & [0.3, 0.4] & \varphi & [0.5, 0.5] \end{pmatrix}.$$

By the proposed approach, we can derive the priority weights of the four alternatives as $w_1 = 0.3078$, $w_2 = 0.1057$, $w_3 = 0.2346$, $w_4 = 0.3519$, which results in a raking $x_4 \succ x_1 \succ x_3 \succ x_2$. Therefore the best alternative is x_4.

If Xia and Xu (2011)'s approach is utilized, the priority weight intervals of the alternatives can be derived as $\tilde{w}_1 = [0.6839, 1.5653]$, $\tilde{w}_2 = [0.9375, 1.0108]$, $\tilde{w}$
$3 = [0.6683, 0.9492]$, $\tilde{w}_4 = [0.9457, 1.6430]$. By the ranking approach for interval numbers (Xu & Da, 2002), the ranking of the alternatives can be obtained $x_4 \succ x_1 \succ x_2 \succ x_3$, which is slightly different from the ranking obtained by the proposed approach. The differences between the two approaches are as follows. Xia and Xu (2011)'s approach estimates the missing elements based on the multiplicative consistency first and then aggregates the individual preference relations into a collective one. Based on the collective uncertain fuzzy preference relation, the priority weight intervals of the alternatives are derived. That is to say, Xia and Xu (2011)'s approach just considers the individual consistency of preference relations and does not take the group consensus into account.

6. Discussions on the proposed model

In this section, we discuss the advantages and the limitations of the proposed decision making model. In general, the proposed model has the following distinct advantages.

(1) This proposal is a new attempt to integrate heterogeneous incomplete uncertain preference relations, which allows decision makers to express their preference information over alternatives more flexibly. Through literature review, we can find that most of the existing studies on GDM with heterogeneous information focus on deriving priority weights from heterogeneous certain preference structure (for instance, Fan & Zhang (2010) and Xu (2011)'s work), and little research has been conducted on heterogeneous GDM with uncertain preference structure. Gao and Peng (2011) proposed to utilize heterogeneous uncertain preference relations for SWOT analysis, but the approach cannot be used to deal with the situation where incomplete uncertain preference relations are provided. Xia and Xu (2011) and Liu et al. (2012) proposed some methods to complement the missing elements for an uncertain fuzzy or multiplicative preference relation and applied them into group decision making. However, these methods need to complement

the missing elements first and cannot deal with GDM problems with heterogeneous incomplete uncertain preference relations. Although Xu and Chen (2008b) investigated group decision making problems with distinct uncertain preference structures including interval utility values, interval fuzzy preference relations and interval multiplicative preference relations, the work focuses on multi-attribute GDM problems and cannot derive priority weights from heterogeneous (incomplete) preference relations as mentioned in this paper.

(2) The basic ideas of the proposed approach are straightforward. Based on the multiplicative consistency of different formats of preference relations, a bi-objective optimization model which aims to minimize both the group consensus index and the individual consistency index (the smaller the two indices, the better the solution) is established to derive the priority weights. Therefore, the obtained result is more reasonable. As the priority weights are obtained by solving optimization models, fewer transformations from heterogeneous preference relations to a single type of preference relation are needed, which can avoid information loss to an extent.

(3) The proposed model can be considered as a general decision making model. If all the elements of the indication matrices are 1, the proposed model can be used to deal with GDM problems with heterogeneous complete uncertain preference relations. Moreover, the proposed model can also be used to deal with GDM problems with any combinations of the four types of preference relations mentioned in this paper.

(4) From the illustrative examples, we can find that the priority weights obtained by the proposed approach lie in the unit interval. Therefore, the proposed approach can be integrated with the Analytic Hierarchy Process (Saaty, 1980) to deal with multi-criteria decision making problems under group decision making environment, especially when heterogeneous incomplete uncertain preference relations are involved in the decision making problem.

However, the proposed models still have some limitations. First of all, if the number of alternatives or decision makers is too large, the number of constraints in the models will be much larger. In this case the resolution may be a little complex. Thus an interesting research topic may be to simplify the models or to investigate other simple decision making models. Second, in order to integrate uncertain linguistic preference relations, we transform them into uncertain fuzzy preference relations, which may lead to some information loss. Finally, the proposed approach just considers four typical formats of uncertain preference structures which cannot integrate preference structures in the form of interval utility values and uncertain preference ordinals. All the issues will be investigated in the future.

7. Conclusions

For practical GDM problems, decision makers usually have different cultures and education backgrounds, as a result, the preference relations provided by them may be of different formats. In addition, due to the lack of knowledge and experience, the preference relations are usually incomplete and uncertain ones. In this paper, we present an approach to addressing group decision making problems with heterogeneous incomplete uncertain preference relations. Based on the multiplicative consistency of different preference relations, the group consensus index and collective individual consistency index are defined. Considering the two indices, a bi-objective optimization model is established to derive the priority weights of alternatives, which is easy to implement on the computer.

Although the illustrative example presented in this paper is an investment alternative selection problem for a multinational corporation, the proposed GDM model can also be applied to deal with other practical decision making problems, such as supplier selection, company performance appraisal and quality function deployment, especially when decision makers are from different countries/areas or with different culture and education backgrounds or when decision makers cannot reach a consensus on which type of preference relations should be used.

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