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Mathematical model for quality cost optimization

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ABSTRACT

Quality engineering uses robust design in order to improve quality by reducing the effects of variability. Variability of the product can be reduced by two stages. One is parameter design which is adjustable to the nominal value so that output is less sensitive to the cause of variability. Other one is tolerance design which is to reduce the tolerance in order to control variability. All costs incurred in a product life cycle can be divided into two categories—manufacturing cost before the sale to the customer and quality loss after the shipment of the product to the customer. It is very important to find the optimum tolerances for each of the characteristics. A balance between manufacturing cost and quality loss should be arrived at in the tolerance design for quality improvement and cost reduction. For the case of Nominal-The-Best, a mathematical model is developed in order to determine the optimum product tolerance and minimize the total cost which includes the manufacturing cost and the quality loss. Since the process capability index (C_{pm}) shows the balance of quality responsibility between the design and the manufacturing engineers, this is taken as the basis in developing the functional relationship between the variability of the product and the tolerance. Based on these relationships, the total cost of model can be expressed as a function of product tolerance from which the optimal tolerance limits can be found out. Finally, using this model a tolerance design approach that increases the quality and reduces the cost can be achieved in the early stages of the product process design stage itself.

1. Introduction

Quality engineering uses robust design in order to improve quality by reducing the effects of variability. Variability of the product can be reduced by two stages [10]. One is parameter design, which is adjustable to the nominal value so that output is less sensitive to the cause of variability. Other one is tolerance design, which is to reduce the tolerance in order to control variability.

All costs incurred in a product life cycle can be divided into two categories—manufacturing cost before the sale to the customer and quality loss after the shipment of the product to the customer [2]. Using parameter-design technique the optimum level of each control factor for the case of Nominal-The-Best quality characteristic is determined. There is no manufacturing cost associated with parameter design i.e., changing of the nominal value of the product parameters.

During the tolerance design, the design engineer will systematically specify the performance levels of certain factors needed to meet the requirement of the quality characteristics. Designers can get the tolerance limit for each factor in order to achieve this design objective.

The loss function is an expression of estimating the cost of quality with respect to the target value and the variability of the product characteristics in terms of monetary loss due to product failure in the hands of the customer [1]. The loss function is a way to show the economic value of reducing the variability and staying very close to the target value. Whereas in the case of manufacturing cost for a product, cost usually increases as the tolerance of the quality characteristic are close to the ideal value [4]. That is why there is a need for more refined and precise operations as the ranges of output are reduced. Therefore, a balance between manufacturing cost and quality loss should be arrived at in the tolerance design for product quality improvement and cost reduction. Since the process capability index (C_{pm}) [3] shows the balance of quality responsibility between the design and the manufacturing engineers, this is considered as a tool for the estimation of the product variability in terms of product tolerance.

If the tolerances are very tight the manufacturing cost will be high and loose tolerances result in low manufacturing cost. The cost equation suggested by Mr. Spotts is $A+B/t^2$ [5], where t is the tolerance. It can be seen that tight tolerance specifications results in more manufacturing cost since additional operations cost, high precision equipment and machines and slower manufacturing

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rates. This tolerance cost equation is considered for the mathematical modeling.

In addition to the manufacturing cost incurred, Dr. Taguchi's [6] quality loss function $L(y)=K\,(y-T)^2$ which is associated with deviation from the target value T, is also considered.

In general even though less manufacturing cost, loose tolerance indicates that the variability of the product characteristic will be high resulting in poor-quality and high-quality loss. On the other hand, a tight tolerance indicates that the variability of the product characteristic will be less, resulting in very good quality reducing quality loss but increasing manufacturing cost. In addition to these two costs, associated scrap/reworks costs are also considered when the quality characteristic falls outside the tolerance limits [9]. Hence the total cost that consists of quality loss and manufacturing cost is applied to find the most economical and efficient way of determining the tolerance limits.

2. Notations

y	output quality characteristic
Δ_o	amount of value deviated from target value
t	product tolerance of y
T	target value of y
μ	process mean of y
N	normal distribution describes the variable y
$\sigma_{(t)}$	product variability which is assumed to be a function of product tolerance
$L(y)$	quality loss function
K	cost coefficient of quality-loss function
C_R	raw material cost
C_I	inspection cost
$C_P(t)$	manufacturing cost (conversion cost) as function of product tolerance 't'
$C_R(t)$	rework cost which is assume to be equal to $C_P(t)$
C_S	scrap costs
$TC(t)$	expected total cost which is a function of product tolerance 't'. It includes quality loss, manufacturing cost, scrap and rework costs
C_p	process capability
C_{pm}	process capability index

3. Description of model

The quality-loss function shows the way to economic value of reducing the variability and reaching closer to the target value. Hence the quality engineers need to establish the design target value for the lowest cost and to reduce the process variability through optimal design. The design of the tolerance limits for a certain characteristic of a part will influence the variability of the manufacturing parts in the measurements of that characteristic. It is very difficult to arrive at the exact relationship between the product tolerance and process variability because of the assumptions made in building the model. An attempt is made to determine the nearest relationship using the process capability index (C_{pm}) [3,7,8].

We know that the process capability $C_P=(U–L)/6\sigma$

$$C_{pm}=\frac{C_p}{\sqrt{1+[(\mu-T)^2/\sigma^2]}}$$

Substituting for C_p from above,

$$C_{pm}=\frac{U-L}{6\sigma\sqrt{1+[(\mu-T)^2/\sigma^2]}}$$

$$=\frac{U-L}{6\sigma\sqrt{\sigma^2+[(\mu-T)^2]/\sigma^2}}$$

$$C_{pm}=\frac{U-L}{6\sqrt{(\mu-T)^2+\sigma^2}} \tag{1}$$

where 'U' and 'L' are upper and lower specification limits, respectively.

The difference between 'U' and 'L' $=2t$, substituting $U–L=2t$ in Eq. (1)

$$C_{pm}=\frac{2t}{6\sqrt{(\mu-T)^2+\sigma^2}}$$

Squaring on both sides

$$C_{pm}^2=\frac{4t^2}{36\left((\mu-T)^2+\sigma^2\right)}$$

$$\sigma^2=\frac{4t^2}{36C_{pm}^2}-(\mu-T)^2$$

$$\sigma=\sqrt{\frac{t^2}{9C_{pm}^2}-(\mu-T)^2}$$

Since the process mean 'μ' can be adjusted to the target value 'T' without causing additional cost or difficulty in a practical performance, the above equation can further be simplified as follows:

$$\sigma=\frac{t}{3C_{pm}}$$

hence

$$t=\frac{3C_{pm}}{\sigma}=P \tag{2}$$

Eq. (2) is applied for the variability estimation in the following model development. After substituting 'σ' in Eq. (2), the only unknown variable in the cost function will be tolerance 't' which needs to be determined so that the total cost will be minimized.

4. Model development

Development of this model is based on the process average (μ) being equal to target value (T), which is also known as nominal value. The quality characteristic has a finite value and the quality loss is symmetric about the target value. The quality-loss function is

$$L(y)=K(y-T)^2 \tag{3}$$

The ideal value for the quality characteristic is located at target 'T' [6]. Hence, the closer the quality characteristic value to the target, the better the product quality will be. The expected value of the quality-loss function can be expressed as

$$E[L(y)]=K[(\mu-T)^2+\sigma^2] \tag{4}$$

where μ is the process mean of 'y'.

Fig. 1 shows the representation of Nominal-The-Best case considering both normal probability distribution function and Taguchi's quality-loss function [2].

Eq. (4) has the following two components:

1. $K(\mu–T)^2$ resulting from the deviation of the average value of y from the target value.
2. $K\sigma^2$ resulting from the mean square deviation of y around its mean.

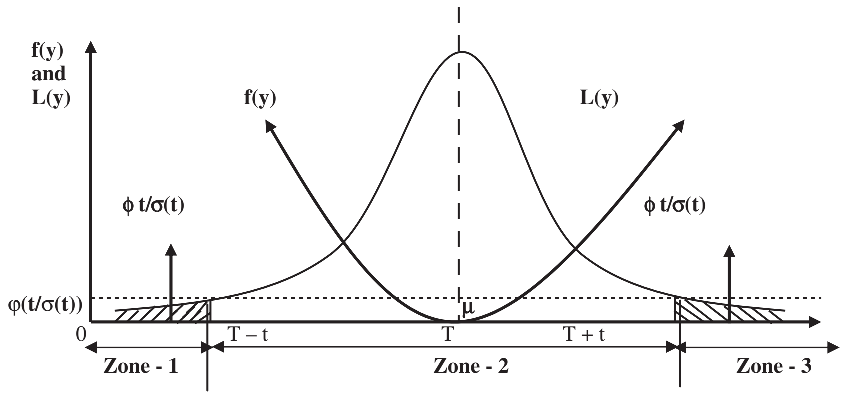

Fig. 1. Shown above represents the normal distribution function with a product tolerance superimposed with the Dr. Taguchi quality-loss function.

Because of the assumption of process mean (μ) equal to the target value (T) for Nominal-The-Best case, Eq. (4) can be expressed as $E[L(y)] = K\sigma^2$.

The following assumptions are made in development of the mathematical model:

- The normal distribution function and the quality-loss function are symmetric about the mean.
- The process mean 'μ' can be adjusted to the target value 'T'.
- The cost of rework is same as that of manufacturing cost.
- The product is re-inspected after the rework.
- The quality characteristic of the product is maintained very close to its target value and hence the reworked components will not have any quality loss.

Fig. 1 has three zones:

1. ∞ to $(T-t)$ is the scrap zone.
2. $(T-t)$ to $(T+t)$ is the acceptable zone.
3. $(T+t)$ to ∞ is the rework zone.

When the product characteristic falls below the lower specification limit ($T-t$) (zone-1), it is treated as a scrap. The cost components involved in this are raw-material cost, manufacturing cost, inspection cost and scrap cost, and the cost of such product is represented as $C_R+C_P(t)+C_I-C_S$.

The cost involved for any component in the acceptable range $(T-t)$ to $(T+t)$ (zone-2) is the summation of raw-material cost, quality loss, manufacturing cost (conversion cost) and inspection cost and is represented as

$$C_R + K(y-T)^2 + C_P(t+)C_I$$

When the product characteristic falls above the upper tolerance limit $(T+t)$ (zone-3), it is assumed that this can be reworked and the additional cost of rework is $C_R(t)$. The cost of product falling in this zone can be represented as $C_R+2C_P(t)+2C_I$.

Hence the total expected cost is given by

$$\mathrm{TC}(y) = \begin{cases} C_R + C_P(t) + C_I - C_S, & y < T - t \\ C_R + C_P(t) + K(y-T)^2 + C_I, & T - t \leqslant y \leqslant T + t \\ C_R + 2C_P(t) + 2C_I, & y > T + t \end{cases}$$

The quality characteristic of a product shipped to the consumers should be with in $(T-t)$ and $(T+t)$ with a truncated distribution. In the case of doubly truncated distribution the normal probability density function is given by

$$f(y) = \begin{Bmatrix} KN(\mu,\sigma) & \text{if } T-t \leqslant y \leqslant T+t \\ & \text{otherwise} \end{Bmatrix} \quad (5)$$

where 'K' is a proportionality constant and is obtained as follows.

Consider

$$\int_{-\infty}^{\infty} f(y)\,\mathrm{d}y = 1$$

i.e.,

$$\int_{T-t}^{T+t} KN(\mu,\sigma)\,\mathrm{d}y = 1$$

Upon integration, the constant

$$K = \frac{1}{[2\Phi(t/\sigma) - 1]}$$

$E(y$ under doubly truncated distribution)

$$= \int_{T-t}^{T+t} \frac{y}{[2\phi(t/\sigma)-1]\sigma\sqrt{2\pi}} \mathrm{Exp}\left\{\frac{-1}{2}\left(\frac{y-T}{\sigma}\right)\right\}^2 \mathrm{d}y \quad (6)$$

$$E(y \text{ under doubly truncated distribution}) = T \quad (7)$$

$V(y$ under doubly truncated distribution)

$$= \int_{T-t}^{T+t} (y-T)^2 \frac{1}{[2\phi(t/\sigma)-1]\sigma\sqrt{2\pi}} \mathrm{Exp}\left\{\frac{-1}{2}\left(\frac{y-T}{\sigma}\right)^2 \mathrm{d}y\right\} \quad (8)$$

$V(y$ under doubly truncated distribution)

$$= \sigma^2\left\{1 - \frac{2(t/\sigma)\varphi(t/\sigma)}{[2\phi(t/\sigma)-1]}\right\} \quad (9)$$

Since expected mean $E(y) = T$, then the expected value of $L(y)$ is

$$E(L(y)) = E(K(y-T)^2) = K[(E(y)-T)^2 + V(y)] = KV(y) \quad (10)$$

The total cost can be expressed as follows:

$$\mathrm{TC}(y) = \begin{cases} C_R + C_P(t) + C_I - C_S, & y < T - t \\ C_R + C_P(t) + K(y-T)^2 + C_I, & T - t \leqslant y \leqslant T + t \\ C_R + 2C_P(t) + 2C_I, & y > T + t \end{cases} \quad (11)$$

Then total expected cost

$$\mathrm{TC}(t) = E(C(y)) = [C_R + C_P(t) + C_I - C_S]\int_{-\infty}^{T-t} N(\mu,\sigma)\mathrm{d}y + \int_{T-t}^{T+t} [C_R + C_P(t) + K(y-T)^2 + C_I]N_t(\mu,\sigma)\mathrm{d}y + [C_R + 2C_P(t) + 2C_I]\int_{T+t}^{\infty} N(\mu,\sigma)\mathrm{d}y$$

Since the normal distribution is symmetric,

$\int_{-\propto}^{T-t} N(\mu,\sigma)dy$ will be equal to $\int_{T+t}^{\propto} N(\mu,\sigma)dy$.

$$TC(t) = [3C_P(t) - C_S + 2C_R + 3C_I]\phi(t/\sigma) + C_P(t) + C_I + C_R + K\sigma 2\left\{1 - \frac{2(t/\sigma)\varphi(t/\sigma)}{2\phi(t/\sigma)-1}\right\}$$

$$TC(t) = (3C_P(t) - C_S + 2C_R + 3C_I)A_1 + C_P(t) + C_I + C_R + K\sigma^2 A_2 \quad (12)$$

where

$$A_1 = \phi(t/\sigma) \quad \text{and} \quad A_2 = \left\{1 - \frac{2(t/\sigma)\varphi(t/\sigma)}{[2\phi(t/\sigma)-1]}\right\}$$

Differentiating Eq. (12) with respect to 't'

$$TC(t)' = 3C_P'(t)A_1 + C_P'(t) + 2KA_2\sigma\sigma' = 0 \quad (13)$$

The sufficient condition is: $TC''(t) = 3C_P''(t)A_1 + C_P''(t) + 2KA_2(\sigma')^2\sigma\sigma''$.

Differentiating Eq. (2) $\sigma = t/P$

we get $\sigma' = 1/P$ which is a constant and hence $\sigma'' = 0$.

Hence the above expression becomes

$$TC''(t) = 3C_P''(t)A_1 + C_P''(t) + 2KA_2(\sigma')^2 > 0 \quad (14)$$

Therefore using Eq. (13), the tolerance (t_o) and the total cost can be found out.

We know that the manufacturing cost equation is $C_P(t) = A+B/t^2$.

Differentiating above equation with respect to 't', we get

$$C_P'(t) = -2B/t^3$$

$\phi(t/\sigma)$ and $\varphi(t/\sigma)$ are also constants.

By substituting these in Eq. (13) we get

$$(-2B/t^3)(3A_1+1)(2KA_2t/P^2) = 0 \quad (15)$$

therefore

$$t_o = \sqrt[4]{(3A_1+1)BP^2/KA_2} \quad (16)$$

which is the optimal solution of total cost TC(t).

5. Validation of the model using real-life data

The following live data are taken from a factory manufacturing compressors for refrigerators, with Nominal-The-Best characteristic: As mentioned earlier manufacturing cost equation is $C_P(t) = A+B/t^2$

For a given design tolerance $t_d = \pm 30$ V, $A = 900$, $B = 90{,}000$.

$\therefore C_p(t) = 900+90{,}000/30^2 = 1000$; $T = 230$ V, $C_R = $ Rs. 500, $C_S = $ Rs. 0; $C_I = $ Rs. 10, $\mu = 230$ V.

$\phi(t/\sigma) = \phi(3) = 0.00135$ and $\varphi(t/\sigma) = \varphi(3) = 0.0044$ (From normal tables).

Estimated consumer loss per unit (A_o) = Rs. 5000.

Using Taguchi quality-loss function $L(y) = Kt^2$.

$A_0 = K\Delta_0^2$, where $L(y) = A_0$ and $\Delta_0 = (y-T) = t$.

Therefore, $K = A_0/\Delta_0^2 = 5000/30^2 = 5.556$.

Substituting in Eq. (16)

$$t_o = \sqrt[4]{(3A_1+1)BP^2/KA_2}$$

$A_1 = 0.00135$ and

$$A_2 = \left\{1 - \frac{2(t/\sigma)\varphi(t/\sigma)}{[2\phi(t/\sigma)-1]}\right\} = 1 - 2(0.00135 \times 0.0044)/(2 \times 0.00135 - 1) = 1.02647$$

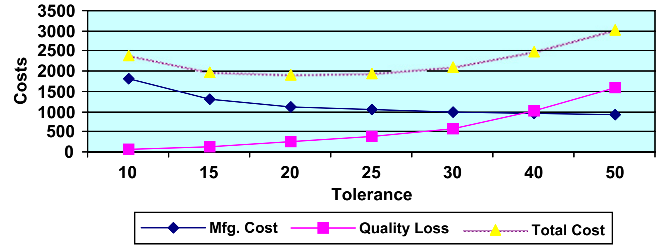

Fig. 2. Costs Vs tolerance.

Table 1
Various costs (Rs.) Vs tolerances (V)

Sl. No.	Tolerance—t	Manufacturing cost—$C_P(t)$	Quality loss $L(y)$	Total cost—TC
1	10	1800	63	2382
2	15	1300	142	1959
3	20	1125	253	1894
4	25	1044	396	1955
5	30	1000	570	2085
6	40	956	1014	2486
7	50	936	1585	3036

$$\therefore t_o = ((3 \times 0.00135 + 1) \times 90000 \times 3^2/5.556 \times 1.02647)^{1/4}$$

i.e., optimum tolerance (t_o) = 19.4 V.

Substituting in Eq. (12) for optimum cost at this tolerance

$$TC(t) = (3C_P(t) - C_S + 2C_R + 3C_I)A_1 + C_P(t) + C_I + C_R + K\sigma^2 A_2.$$

Manufacturing cost at this optimum tolerance

$$C_P(t) = 900 + B/t^2 = 900 + 90{,}000/19.4^2 = Rs. 1139.13$$

$$TC_0 = (3 \times 1139.13 - 0 + 2 \times 500 + 3 \times 10)0.00135 + 1139.13 + 10 + 500 + 5.556(19.4/3)^2 \times 1.02647 = \text{Rs. } 1893$$

The total cost at the specified tolerance t_d at 30 V is

$$TC(t_d) = (3 \times 1000 - 0 + 2 \times 500 + 3 \times 10)0.00135 + 1000 + 10 + 500 + 5.556(30/3)^2 \times 1.02647$$

$TC(t_d) = $ Rs. 2085.

The manufacturing cost $C_P(t)$, the cost of quality loss $L(y)$ and the total cost (TC) against various tolerances (Fig. 2) are calculated in similar manner and are presented in Table 1.

It is seen from above graph that as the tolerance is narrowed down, the manufacturing cost is increasing, the cost of quality loss to the society is decreasing and total cost is decreasing up to a certain value of tolerance and increases further there on.

6. Conclusions

From the literature review, it has been observed that there are two parallel developments for determining the optimum tolerances, one based on the manufacturing cost without considering the quality loss, and the other one based on the quality loss without considering the manufacturing cost. Hence, an attempt is made to determine the optimum tolerance by combining these two costs (manufacturing cost and quality loss). The process capability index C_{pm} is taken as a tool in building the mathematical model to arrive at the optimal tolerance and the minimum cost. A mathematical model has been developed for Nominal-The-Best quality characteristic for the variable data. This model has been validated considering real life data. This model is very generic in nature that can be applied to any variable characteristic after optimizing the parameter design. Finally, using this model, a tolerance design approach which increases the quality and reduces the cost can be achieved in the early stages of product/process design.

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