FINITE ELEMENT IMPLEMENTATION OF FRICTIONAL PLASTICITY MODELS WITH DILATION

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ABSTRACT

A frictional perfect plasticity model based on a yield criterion proposed by Matsuoka is described. Two methods are proposed to derive the plastic strain rates in such a way that the dilation angle may be specified as an independent parameter; each of these two procedures are described in detail. The proposed frictional plasticity model is suitable for use in one- two- or three-dimensional numerical calculations but special attention is paid to the use of the plasticity model in plane strain finite element computations. For each of the two methods used to derive the plastic strain rates a set of relationships are derived between the parameters used in the proposed plasticity model and the parameters generally specified in plane strain calculations based on non-associated Mohr-Coulomb plasticity models. A comparison is also made between the well known stress dilatancy rule proposed by Rowe and the proposed plasticity model for the case of triaxial compression. The proposed plasticity model has been implemented in a plane strain finite element computer program and a description is given of the salient features of this formulation. The results of a typical footing collapse problem are used to illustrate the application of this numerical model.

INTRODUCTION

Frictional plasticity models of soil behaviour are used extensively in the finite element analysis of problems in soil mechanics. Many of the constitutive models currently used in the analysis of frictional soil are based on a plasticity model in which yield is defined by the Mohr-Coulomb yield criterion. This yield surface, however, contains discontinuities at which the yield function is not differentiable which is a disadvantage if the plastic strain rates are derived from a plastic potential of the same form as the yield function. These singularities are physically unrealistic and also give rise to constitutive models that are difficult to implement numerically unless a numerical procedure to round off the corners of the yield surface is employed.

The development of a plasticity formulation suitable for the constitutive modelling of frictional material generally involves the consideration of two separate issues, namely the choice of a suitable yield surface and the development of a procedure to calculate the plastic strain rates. In this paper, a frictional plasticity model based on a yield criterion proposed by Matsuoka[1] is described. The Matsuoka yield surface may be expressed as a cubic function of the stresses in which no singularities exist (except at the origin) and is therefore well suited as the basis of a plasticity model for use in finite element formulations. A well accepted procedure to derive the plastic strain rates in a model based on the Matsuoka yield function, however, has yet to be developed. This paper describes a frictional plasticity model in which two separate procedures for the derivation of the plastic strain rates are proposed. Both of these approaches have been implemented in a plane strain finite element computer program but a detailed comparison with experimental test data obtained for granular material has yet to be carried out.

The proposed plasticity model is suitable for use in one- two- or three-dimensional numerical calculations but the application of the model to two-dimensional plane strain finite element calculations is described in detail. In order to compare the results of plane strain computations based on this proposed plasticity model with results obtained from other plane strain models it is necessary to relate the parameters used in the proposed plasticity model to the plane strain friction and dilation angles. A detailed description is given of the derivation of these correlations.

Compressive stresses are taken to be positive throughout the paper.

THE MATSUOKA YIELD FUNCTION

The Matsuoka[1] yield function may be written as a function of the principal stresses:-

$$f(\sigma_{ij}) = \frac{(\sigma_2 - \sigma_3)^2}{\sigma_2\sigma_3} + \frac{(\sigma_1 - \sigma_3)^2}{\sigma_1\sigma_3} + \frac{(\sigma_1 - \sigma_2)^2}{\sigma_1\sigma_2} - 8\tan^2\phi_{tc} \qquad (1)$$

where ϕ_{tc} is defined as the triaxial compression friction angle. An alternative expression for the yield surface may be written in terms of stress invariants:-

$$f(\sigma_{ij}) = I_1 I_2 - I_3 \zeta \qquad (2)$$

where $\zeta = 9 + 8\tan^2\phi_{tc}$ and :-

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3$$
$$I_2 = \sigma_2\sigma_3 + \sigma_1\sigma_3 + \sigma_1\sigma_2$$
$$I_3 = \sigma_1\sigma_2\sigma_3$$

This yield surface is closely related to the Mohr-Coulomb yield envelope; a comparison between sections of these two yield loci in the 'π' plane for the special case that they coincide at the 'corners' of the Mohr-Coulomb surface is given in Fig. 1.

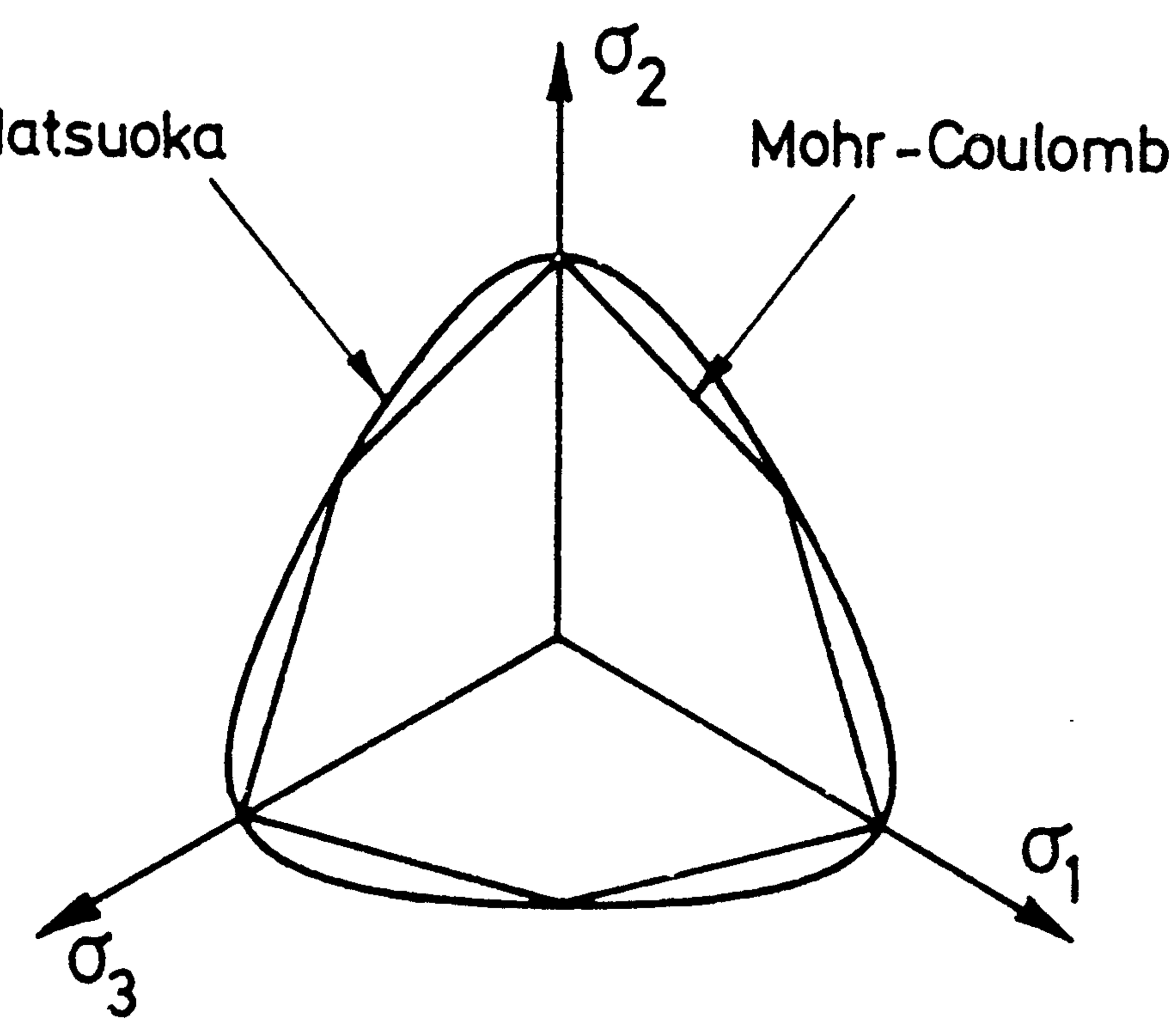

Fig.1 Comparison between the Mohr-Coulomb and Matsuoka yield surfaces

FRICTIONAL PLASTICITY MODEL - APPROACH 1

1. Derivation of plastic strain rates

Two methods are proposed to derive a method by which a plasticity formulation may be developed, based on the Matsuoka yield function, in such a way that the dilation and friction angles may be specified independently of each other. In the first approach, a procedure is used to interpolate between plastic strain rates corresponding to two limits of behaviour; case 'A' in which the flow rule is fully associated and case 'B' in which the flow rule is associated in the 'π' plane with zero plastic volumetric strain rate.

For case 'A', the plastic strain rates are derived from the fully associated plastic potential, $g(\sigma_{ij})$:-

$$g(\sigma_{ij}) = I_1 I_2 - I_3 \zeta \qquad (3)$$

The plastic strain rates are derived from the flow rule:-

$$\dot{\varepsilon}^p_{ij} = \lambda \frac{\partial g}{\partial \sigma_{ij}} \qquad (4)$$

where λ is a positive scalar multiplier and $\dot{\varepsilon}^p_{ij}$ are the plastic strain rates.

For case 'B', the plastic strain rates are derived from the deviatoric terms of the fully associated plastic potential. In this case, the plastic strain rates are given by:-

$$\dot{\varepsilon}^p_{ij} = \lambda \left(\frac{\partial g}{\partial \sigma_{ij}} - \frac{1}{3} \delta_{ij} \frac{\partial g}{\partial \sigma_{kk}} \right) \qquad (5)$$

where the repeated suffices imply summation, δ_{ij} is the Kronecker delta and $g(\sigma_{ij})$ is the fully associated plastic potential.

The material behaviour for the case when the dilation rate lies between these two extremes is obtained by taking a weighted average of the plastic strain rates that give full and zero dilation. The plastic strain rates derived in this way are:-

$$\dot{\varepsilon}^p_{ij} = \lambda \left(\frac{\partial g}{\partial \sigma_{ij}} - \frac{(1 - \gamma_a)}{3} \delta_{ij} \frac{\partial g}{\partial \sigma_{kk}} \right) \qquad (6)$$

The variable γ_a is referred to as the 'degree of association' and is the independent parameter that controls the dilation characteristics of the model. For the case when γ_a is unity the behaviour reduces to the fully associated case, and when γ_a is zero, the dilation rate is zero.

2. Correlations with Plane Strain Parameters

If this frictional plasticity model is used as the basis of a plane strain finite element formulation then it is highly desirable to relate the parameters used in the proposed plasticity model to the plane strain friction and dilation angles. These correlations are necessary in order to compare the results of calculations made using the proposed model with those based on material models for which plane strain parameters are specified directly.

The plane strain friction angle, ϕ_{ps}, and the plane strain dilation angle ψ_{ps} are defined:-

$$\tan^2\phi_{ps} = \frac{(\sigma_1 - \sigma_3)^2}{4\sigma_1\sigma_3} \tag{7}$$

$$\sin\psi_{ps} = \frac{\dot{\varepsilon}_1^p + \dot{\varepsilon}_3^p}{\dot{\varepsilon}_1^p - \dot{\varepsilon}_3^p} \tag{8}$$

where σ_1 and σ_3 are the in-plane principal stresses and $\dot{\varepsilon}_1^p$ and $\dot{\varepsilon}_3^p$ represent the in-plane plastic strain rates. In order to find the relationship between the triaxial compression and plane strain friction angle for case 'A', it is necessary to consider the limiting case for which the elastic strain rates are negligible in comparison with the plastic strain rates. This implies that the out-of-plane plastic strain rate is zero. An expression for the out-of-plane plastic strain rate may be derived from equation (4) which, if set to zero, gives the relationship:-

$$\sigma_2 = (\sigma_1\sigma_3)^{1/2} \tag{9}$$

where σ_2 is the out-of-plane principal stress. If equations (7) and (9) are substituted into the Matsuoka yield function then the following relationship is obtained:-

$$2\sec^2\phi_{tc} = \sec\phi_{ps} + \sec^2\phi_{ps} \tag{10}$$

For case 'B', the out-of-plane stress for the limiting case of full plasticity may be shown to be:-

$$\sigma_2^2 = \frac{\sigma_1^2\sigma_3^2}{\sigma_1^2 + \sigma_3^2} \tag{11}$$

The triaxial compression and plane strain friction angles in this case are related by the expression:-

$$\sec^2\phi_{ps}\left[1 + \frac{1}{[1 + \sin^2\phi_{ps}]^{1/2}}\right] = 2\sec^2\phi_{tc} \tag{12}$$

In general, it is necessary to correlate the plane strain friction and dilation angles with the plasticity parameters used in the proposed model for the case where the degree of association lies between the limits of zero and full dilation. This correlation is, again, derived by setting the out-of-plane plastic strain rate to zero which, from equation (6), gives the expression:-

$$\frac{\partial g}{\partial \sigma_2} = \frac{(1 - \gamma_a)}{3}\left(\frac{\partial g}{\partial \sigma_1} + \frac{\partial g}{\partial \sigma_2} + \frac{\partial g}{\partial \sigma_3}\right) \tag{13}$$

In order to obtain the required relationships between the plasticity parameters and the plane strain angles, it is necessary to solve simultaneously equations (1), (7), (8) and (13). In this general case it is not possible to express the correlations as a closed form solution; it is instead necessary to derive consistent sets of parameters using a numerical method. Some selected correlations are given in Table 1

Table 1 Correlations between frictional plasticity model parameters and plane strain parameters.

$\phi_{ps} = 30°$	γ_a	ϕ_{tc}	ψ_{ps}
	0.0	27.15°	0.0°
	0.2	26.87°	6.24°
	0.4	26.64°	12.19°
	0.6	26.46°	18.02°
	0.8	26.34°	23.89°
	1.0	26.29°	30.00°

$\phi_{ps} = 40°$	γ_a	ϕ_{tc}	ψ_{ps}
	0.0	37.02°	0.0°
	0.2	36.55°	8.45°
	0.4	36.13°	16.33°
	0.6	35.77°	23.96°
	0.8	35.51°	31.69°
	1.0	35.39°	40.00°

FRICTIONAL PLASTICITY MODEL - APPROACH 2

1. Derivation of plastic strain rates

An alternative approach is proposed in which the plastic strain rates are derived from the flow rule:-

$$\dot{\varepsilon}_{ij}^p = \lambda\frac{\partial g^*}{\partial \sigma_{ij}} \tag{14}$$

where $g^*(\sigma_{ij})$ is the plastic potential:-

$$g^*(\sigma_{ij}) = I_1^* I_2^* - I_3^* \zeta^* \quad (15)$$

and:-

$$\zeta^* = 9 + 8\tan^2\psi_{tc}$$
$$I_1^* = \sigma_1' + \sigma_2' + \sigma_3'$$
$$I_2^* = \sigma_2'\sigma_3' + \sigma_1'\sigma_3' + \sigma_1'\sigma_2'$$
$$I_3^* = \sigma_1'\sigma_2'\sigma_3'$$
$$\sigma_{ij}' = \sigma_{ij} + k\delta_{ij}$$

This plastic potential function is of a similar form to the Matsuoka yield function; the triaxial friction angle is replaced by the triaxial dilation angle, ψ_{tc}, and the apex of the surface is moved from the origin to the point in principal stress space with the coordinates (-k,-k,-k). The parameter k is calculated on the basis that the plastic potential and the yield function must coincide at the current stress state. This condition is used to derive a cubic equation in k from which the required root may be selected.

2. Correlations between Triaxial and Plane Strain Parameters

It is of interest to consider the relationships between ϕ_{tc} and ψ_{tc} and the parameters defined in equations (7) and (8) for the case where the proposed plasticity model is used under conditions of plane strain. Firstly, it is instructive to consider the two limiting cases, i.e. full association and zero dilation, for which it is possible to derive closed form expressions for these relationships.

The case of full association (where the yield function and plastic potential are identical) corresponds to case 'A' described in the previous section; the plane strain and triaxial compression friction angles in this case are therefore related by equation (10).

The case of zero plastic volumetric strain rate is obtained in the limit as k tends to infinity in equation (15). In this case, in the limit that the elastic strain rates are negligible in comparison with the plastic strain rates, the out-of-plane principal stress may be shown to be:-

$$\sigma_2 = \frac{\sigma_1 + \sigma_3}{2} \quad (16)$$

The triaxial and plane strain friction angles are related by the expression :-

$$\tan^2\phi_{tc} = \frac{3}{4}\tan^2\phi_{ps} \quad (17)$$

In the general case, for a triaxial dilation angle intermediate between zero and the triaxial compression friction angle, it is necessary to resort to a numerical method to correlate the plane strain and triaxial compression friction and dilation angles. Some selected correlations are given in Table 2.

Table 2 Correlations between frictional plasticity model parameters and plane strain parameters.

ϕ_{ps} = 30°	ψ_{tc}	ϕ_{tc}	ψ_{ps}
	0.05°	26.56°	0.06°
	4.78°	26.48°	5.52°
	10.01°	26.42°	11.54°
	15.84°	26.34°	18.21°
	19.72°	26.31°	22.62°
	26.29°	26.29°	30.00°

ϕ_{ps} = 40°	ψ_{tc}	ϕ_{tc}	ψ_{ps}
	0.06°	36.00°	0.07°
	6.15°	35.83°	7.10°
	12.94°	35.67°	14.90
	20.66°	35.52°	23.69°
	25.96°	35.45°	29.63°
	35.39°	35.39°	40.00°

3. Consistency of the Frictional Plasticity Model with Rowe's Stress Dilatancy Rule for Triaxial Compression

It is of interest to determine the relationship between ϕ_{tc} and ψ_{tc} for which the proposed plasticity model matches the stress dilatancy rule proposed by Rowe[2], for the case of triaxial compression. This stress dilatancy rule may be written in the general form:-

$$R = K\,D \quad (18)$$

where, for the case of triaxial compression (i.e. $\sigma_2 = \sigma_3$ and $\sigma_1 > \sigma_2$),:-

$$R = \frac{\sigma_1}{\sigma_3} \quad (19)$$

$$D = 1 - \frac{\dot{\varepsilon}_1^p + 2\dot{\varepsilon}_3^p}{\dot{\varepsilon}_1^p} \qquad (20)$$

$$K = \frac{1 + \sin\phi_{cv}}{1 - \sin\phi_{cv}} \qquad (21)$$

where ϕ_{cv} is the critical state friction angle.

For the case of triaxial compression, the Matsuoka yield function may be used to give an expression for the stress ratio, R,:-

$$R = \frac{1 + \sin\phi_{tc}}{1 - \sin\phi_{tc}} \qquad (22)$$

The flow rule given in equation (14) may be used to derive an expression for the dilation parameter, D. If these expressions for R and D are substituted into Rowe's stress dilatancy rule (equation (18)) then it is possible to derive the following condition for the proposed plasticity model to be consistent with Rowe's stress dilatancy rule:-

$$\sin\psi_{tc} = \frac{\sin\phi_{tc} - \sin\phi_{cv}}{1 - \sin\phi_{tc}\sin\phi_{cv}} \qquad (23)$$

It is of interest that this simple relationship between triaxial compression, dilation and friction angles is of the same form as the relationship obtained by matching Rowe's stress dilatancy rule in plane strain using the non-associated Mohr-Coulomb plasticity model[2].

Rowe's stress dilatancy rule for triaxial compression is therefore consistent with the proposed plasticity model provided that the triaxial dilation angle is derived from equation (23).

FINITE ELEMENT IMPLEMENTATION

The proposed plasticity model has been implemented in a plane strain linear elastic-perfectly frictional finite element formulation. In this formulation, the strain rates are decomposed into elastic and plastic components:-

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ij}^e + \dot{\varepsilon}_{ij}^p \qquad (24)$$

where $\dot{\varepsilon}_{ij}^e$ is the elastic component and $\dot{\varepsilon}_{ij}^p$, the plastic component, is zero if the stress point lies inside the yield surface and given by either equation (6) or equation (14) (depending on which approach is being used for the plastic strain rate derivation) when the stress point lies on the yield surface.

The stress rate $\dot{\sigma}_{ij}$ is related to the strain rate by the constitutive equation:-

$$\dot{\sigma}_{ij} = \left(D_{ijkl}^e + D_{ijkl}^p\right)\dot{\varepsilon}_{kl} \qquad (25)$$

where D_{ijkl}^e are the elastic material constants. The parameters D_{ijkl}^p are zero unless the stresses lie on the yield surface in which case they are given by:-

$$D_{ijkl}^p = \frac{D_{ijmn}^e f_{mn} \dot{\varepsilon}_{op}^p D_{opkl}^e}{f_{qr} D_{qrst}^e \dot{\varepsilon}_{st}^p} \qquad (26)$$

where:-

$$f_{ij} = \frac{\partial f}{\sigma_{ij}} \qquad (27)$$

and the plastic strain rates are derived either from equation (6) or equation (14).

The solution procedure used in the finite element formulation is based on the use of the 'Modified Euler' procedure proposed by Sloan[3]. A feature of this solution scheme is that it is necessary to integrate the constitutive equation over each calculation increment in order to update the stresses at the Gauss points at the end of each stage of the calculation. This integration is performed numerically using an error control procedure described by Sloan[4]. A full description of the finite implementation of this constitutive model (for the case where the plastic strain rates are derived using equation (6)) is given by Burd[5].

The calculation of the pressure-displacement response of a smooth plane strain footing on weightless frictional soil with a constant vertical surcharge applied to the surface is described below as an example of a typical application of this finite element formulation. The mesh used in the finite element calculation consists of six-noded triangular elements and is plotted in Fig. 2. The Poisson's ratio of the soil is taken to be 0.35 and the plasticity model is based on the parameters $\phi_{tc} = 26.46°$ and $\gamma_a = 0.6$. (Note from Table 1 that these plasticity parameters correspond to a plane strain friction angle of 30°). At the start of the finite element calculation, the vertical and horizontal stresses are taken to be constant within the soil and equal to the surcharge stress. The footing pressure-displacement response obtained from the finite element analysis is plotted in

Fig. 3 where p represents the footing pressure, q is the surcharge applied to the soil surface, δ is the footing displacement, B is the footing half-width and E is the soil Young's modulus.

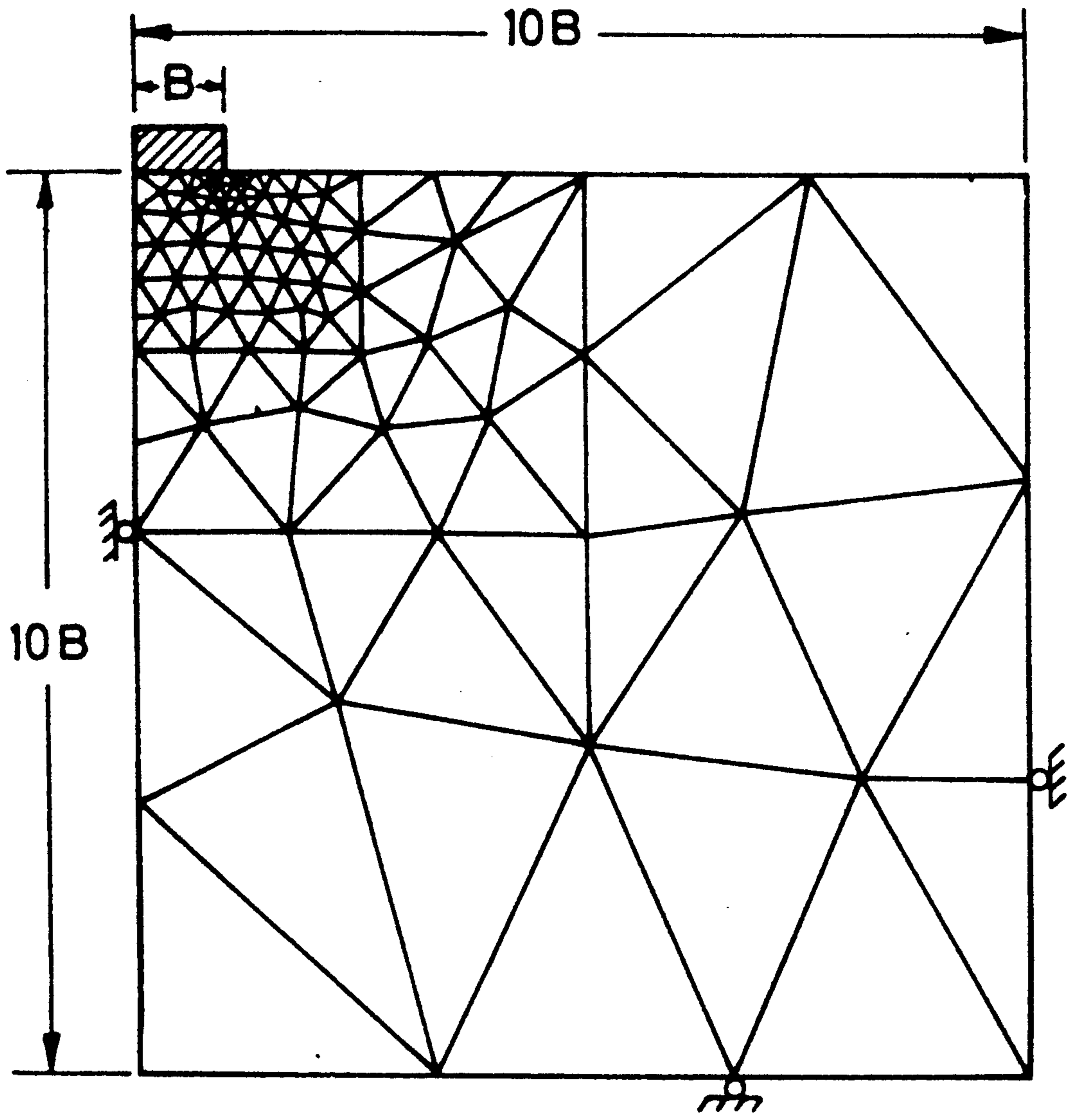

Fig. 2 Finite element mesh

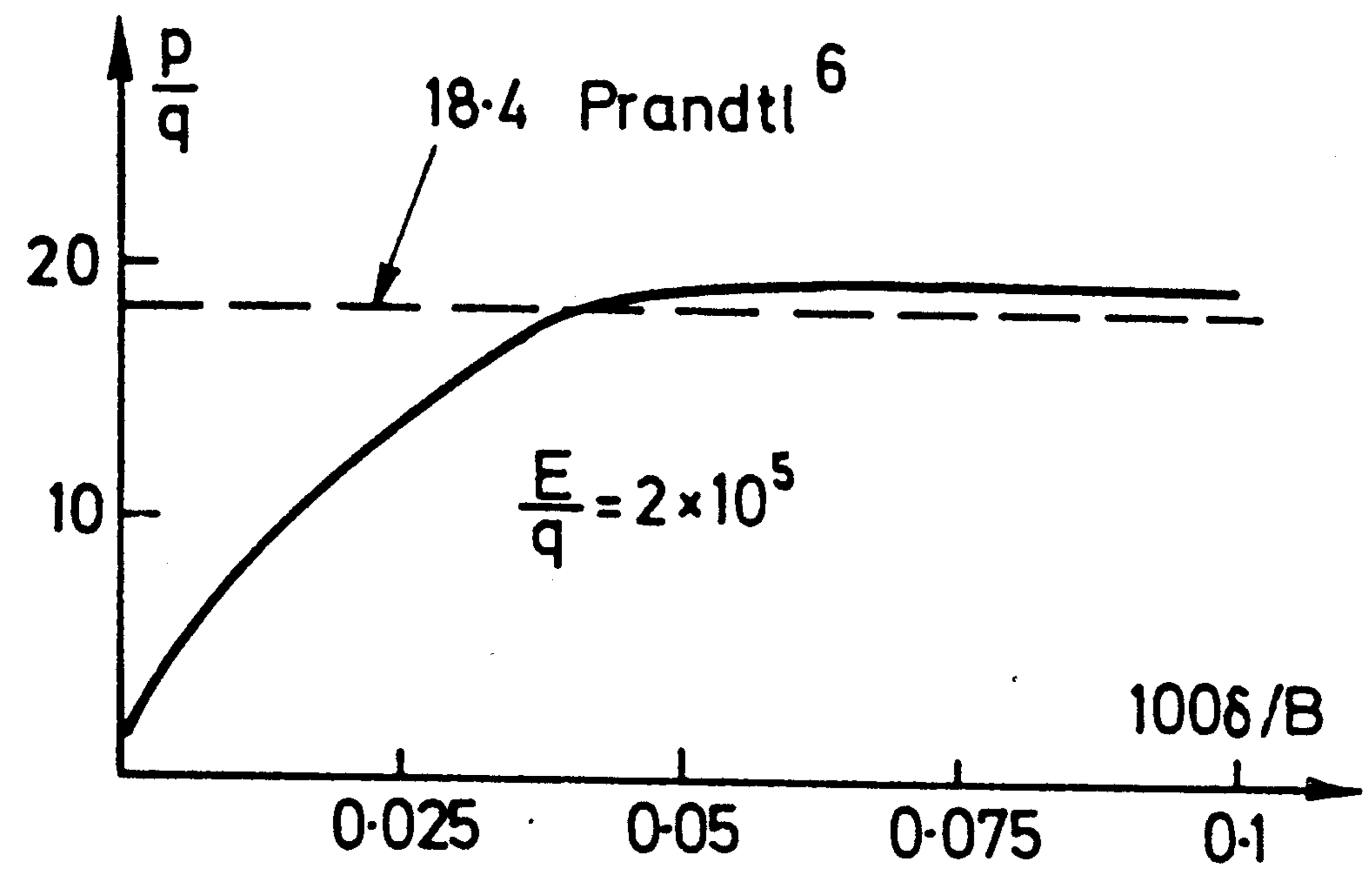

Fig. 3 Pressure-displacement response for a smooth footing on weightless soil

An exact solution to this problem is given by Prandtl[6]:-

$$p_f = q \tan^2\left(\frac{\pi}{2} + \frac{\phi_{ps}}{2}\right)\exp\left(\pi\tan\phi_{ps}\right) \qquad (28)$$

where p_f is the footing pressure at collapse. The Prandtl solution is compared with the finite element results in Fig. 3.

CONCLUSION

The Matsuoka and Mohr-Coulomb yield surfaces are closely related; the Matsuoka surface has the advantage from the computational point of view, however, that it is described by a single cubic function of the stresses which is everywhere differentiable, except at the origin. The Matsuoka yield function is a good mathematical basis for a frictional plasticity model but a well established procedure for the derivation of the plastic strain rates is not yet available. A frictional plasticity model is described in the paper in which two alternative procedures are proposed to derive the plastic strain rates. The proposed model has not yet been compared with suitable experimental data; further work in this area is needed.

The relationship between the friction and dilation angles necessary for the proposed plasticity model to match Rowe's stress dilatancy rule in triaxial compression is given in equation (23). It is of significance that this relationship is of the same form as that derived by Rowe[2] for the dilatancy rule to match a plane strain non-associated Mohr-Coulomb plasticity model. This observation suggests that the proposed plasticity model is a logical extension of the plane strain non-associated Mohr-Coulomb model to the case of triaxial compression.

The frictional plasticity model described in this paper is not limited to the case of perfect plasticity. The model has been developed to include the possibility of the triaxial friction angle varying according to a suitable hardening law with the triaxial dilation angle linked to the other material parameters by equation (23). Full discussion of this extended model, however, is beyond the scope of this paper.

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